Geometric criterion for Pisot numbers with an application to a theorem of Brauer and Hurwitz

by

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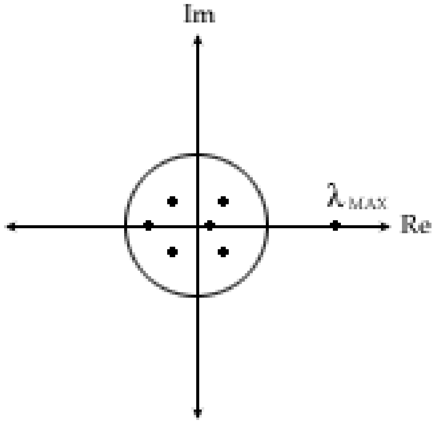
Abstract

We describe the geometric criterion for Pisot–Vijayaraghavan numbers, hereafter referred to simply as Pisot numbers, through analysis of the mapping of an infinite “trapping slab” into its interior by iteration of a matrix with Pisot characteristic polynomial. In particular, we demonstrate that the simplest form of this method restricted to the companion matrices of suitable arbitrary polynomials reduces to the Pisot conditions proven by Brauer and Hurwitz [1, p. 251].

1. Introduction.

A *Pisot number* is an algebraic real integer greater than one such that its conjugate elements are of modulus less than one. Notice that the definition of a Pisot number actually follows from the definition of a class of polynomials which we shall refer to as Pisot polynomials. Then a *Pisot polynomial* is a monic polynomial with integer coefficients such that one root is positive, real, simple, and greater than one, and all other roots are of magnitude less than one (**Fig. 1**). We can extend the idea of Pisot entities to include matrices by recognizing that a Pisot polynomial is the characteristic polynomial for an infinite number of matrices. Thus we can say that an $n \times n$ matrix is a *Pisot matrix* iff its characteristic polynomial is Pisot.

A famous example of a nontrivial Pisot number is the golden mean $\frac{1+\sqrt{5}}{2}$, whose minimal generating Pisot polynomial is $p(x) = x^2 - x - 1$. An example of a corresponding Pisot matrix is the companion matrix to $p(x)$, $B = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$.



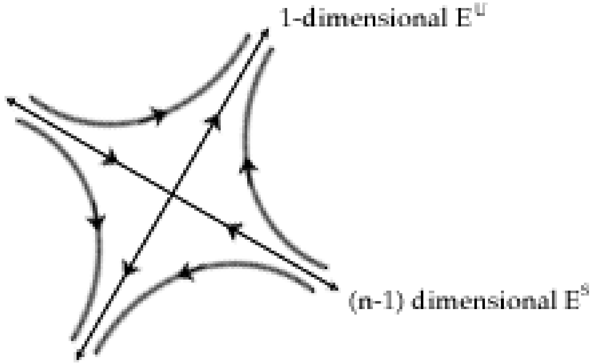


Fig. 1 – Distribution of roots for a Pisot polynomial (or eigenvalues for a Pisot matrix).
Fig. 2 – Dynamics of the linear transformation induced by a Pisot matrix.

The significance of Pisot numbers goes beyond number theory as they play a role in many other areas including harmonic analysis, theoretical computer science, dynamical systems, and the theory of quasi-crystals. In particular, the nonnegative Pisot matrices that we shall study appear as the abelianizations of Pisot substitutions on finite alphabets. The construction of examples of such substitutions relies on a supply of Pisot matrices and these, despite their simple definition, are not trivial to find. [2]

The goal of this work is to develop a method for uncovering Pisot matrices, and consequently Pisot numbers, based on a testing for the characteristic dynamics of the map induced by the matrix. The method, as embodied by Theorems 1 and 2 ahead, enables one to confirm that a given integer nonnegative matrix is Pisot by an algorithm that uses integer arithmetics only (no floating point operations are utilized to determine eigenvalues, etc.). We also deduce, as the simplest case, a classical result due to Hurwitz and Brauer (Theorem 3 ahead) to the effect that a polynomial of the form $p(x) = x^n - b_n x^{n-1} - \ldots - b_2 x - b_1$, where $\forall i \leq n, b_i \in \mathrm{Z}$ and $0 < b_1 \leq b_2 \leq \ldots \leq b_n$, is Pisot. This gives a clear geometric reason why the theorem is true.

2. "Trapping" Slab Method.

Since the roots of an $n \times n$ matrix's characteristic polynomial are equal to its eigenvalues, a Pisot matrix is distinguishable by its spectrum. This spectrum in turn is manifested by specific dynamics when the matrix is viewed as a linear transformation on $\mathbb{R}^n$ (note that in this paper $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}^n$ is the n-dimensional cartesian space). The matrix's $n-1$ eigenvalues of modulus less than one correspond to an $n-1$ dimensional stable invariant subspace, with only the Pisot eigenvalue creating a one dimensional unstable eigenspace. Thus a Pisot matrix will exhibit characteristic saddle dynamics (**Fig. 2**) [3, Prop 1.2.8, p.24 ; 5, p.122]. It is these special dynamics that will allow us to determine whether or not a matrix is Pisot utilizing the following concept of a "trapping slab". Also note that, from the dynamical characterization of a Pisot matrix, $\forall j \in N,\ A$ is Pisot iff A^j is Pisot.

(H) Suppose we have a primitive $n \times n$ matrix $A \geq 0$. For now, we will assume that A is nonsingular and nontrivial in the sense that the eigenvalue of greatest modulus, λ_1, is real, simple, and $\lambda_1 > 1$. We will also suppose that its corresponding eigenvector, v_1, is strictly positive.

Then there exists a corresponding one-dimensional unstable eigenspace $E^U = Span\{v_1\}$. Under hypothesis (H), the dynamics of A may exhibit more unstable directions that just E^U; however, E^U is the direction stretched the most and we have the following consequence of the Perron-Frobenius Theorem [5, p.125]. Define $\prec$ to mean "strictly interior to, except possibly at the origin".

LEMMA 0. If C is a cone in the positive orthant such that E^U passes through the interior of C, then $\exists\ k$ such that $A^k C \prec C$.

Now, let us imagine taking some $n-1$ dimensional simplex, S, and making it the cross-section of an infinitely long "slab", $T = S + E^U$, containing E^U in its interior. The following lemma demonstrates how such a "slab" can act as a "trap" by mapping strictly into its interior region under iterates of A (**Fig. 3**) and that this is in fact equivalent to A being Pisot. It is important to notice that more than one iteration of A may be required to observe the "trapping" effect. Although we shall not make use of the fact, it is also noteworthy that the simplicial nature of the cross-section of the slab we employ is not that critical and we could use slabs with less regular cross-sections (depicted in **Fig. 3**).

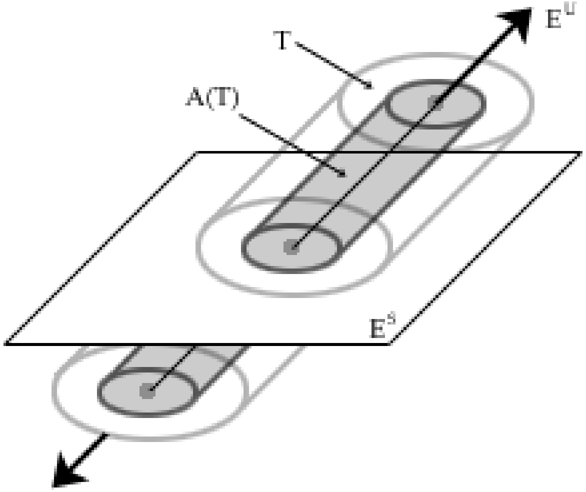


Fig. 3 – "Trapping" effect on an infinitely long "slab" T by application of Pisot matrix A.

Note that in this paper, indices (i,j,k, etc.) are assumed to be positive integers unless stated otherwise. Define the relation $\precneqq$ to mean "strictly interior to".

LEMMA 1. Suppose $T = S + E^U$ is a slab where S is an $n-1$ dimensional simplex and $E^U \precneqq T$. Then A is Pisot iff $\exists j$ such that $A^j T \precneqq T$.

PROOF. ($\Leftarrow$) Suppose $\exists j$ such that $A^j T \precneqq T$. Define

$$\Delta := T \cap E^S .$$

Then Δ is the cross-section of T along E^S and $T = \Delta + E^U$. Now define

$$\Delta_\infty := \bigcap_{i \geq 0} A^{ij} \Delta .$$

Notice that $\forall i \geq 0,\ A^{(i+1)j}\Delta \subseteq A^{ij}\Delta$ by our hypothesis. Since $A^{ij}\Delta$ is a convex polyhedron $\forall i \geq 0$, all intersections of them, including Δ_∞, must also be convex polyhedrons. Then Δ_∞ can be expressed as the convex hull of its vertices, $\Delta_\infty = Conv(p_1,\ldots,p_m)$, where $\forall i \leq m,\ p_i \in E^S$ and $m \leq n$.

CLAIM. $A^j\Delta_\infty = \Delta_\infty$.

PROOF. ($\Rightarrow$) Let $x \in \Delta_\infty$. Then $\forall i \geq 0,\ \exists\, x_i \in \Delta$ such that $x = A^{ij}x_i$. Since $A^j\Delta \subseteq \Delta$, then $A^jx \in \Delta$. Then $A^jx = A^{j+ij}x_i = A^{(i+1)j}x_i$. Thus $A^jx \in \Delta_\infty$ and $A^j\Delta_\infty \subset \Delta_\infty$.

($\Leftarrow$) Let $x \in \Delta_\infty$. Then $\forall i \geq 0,\ \exists\, x_i \in \Delta$ such that $x = A^{ij}x_i$. Equivalently, since A is nonsingular, $x_i = A^{-ij}x$. In particular, $x_1 = A^{-j}x$. Then $x_1 = A^{-j}x = A^{-j}(A^{ij}x_i) = A^{(i-1)j}x_i$. Thus $x_1 \in \Delta_\infty$ and $\Delta_\infty \subset A^j\Delta_\infty$. ■

CLAIM. Δ_∞ is a single point. In particular, $\Delta_\infty = \{0\}$.

PROOF. Suppose not, that is $\exists\, p_m, p_n$ (vertices of Δ_∞) such that $p_m \neq p_n$. Since $A^j\Delta_\infty = \Delta_\infty$, A^j permutes the vertices of Δ_∞. Then $\exists\, l > 0$ such that $A^{jl}p_m = p_m$ and $A^{jl}p_n = p_n$. Since A^{jl} is linear, this implies that the line L passing through p_m and p_n, must be a line of fixed points. In particular, any point p_L of L that lies on the boundary of Δ would satisfy $A^{jl}p_L = p_L$. This is a contradiction since our hypothesis $A^jT \mathrel{\bar{\prec}} T$ implies $A^j\Delta \mathrel{\bar{\prec}} \Delta$. Thus Δ_∞ must be a single point. Since $0 \in \Delta_\infty$, then $\Delta_\infty = \{0\}$. ■

Suppose that A is not Pisot. Then for A restricted to E^S there must exist an eigenvalue η with $|\eta| \geq 1$. Then $\exists\, v \in E^S, v \neq 0$ such that $A^{nj}v \not\to 0$ as $n \to \infty$ by Real Jordan Canonical Form [5, p.97]. Since Δ is a neighborhood of 0, $\exists\, c \in \Re^+$ such that $cv \in \Delta$. Then $cA^{nj}(v) = A^{nj}(cv) \in A^{nj}\Delta \to \Delta_\infty = \{0\}$ by our previous claim, but this is a contradiction with $A^{nj}v \not\to 0$.

($\Rightarrow$) Suppose A is Pisot. Then A restricted to E^S has all eigenvalues of modulus less than one. Then $\forall v \in E^S, A^n v \to 0$ as $n \to \infty$. In particular, $\exists\, j$ such that $\forall i \le n, A^j q_i \mathrel{\bar{\prec}} \Delta$ where q_i is the ith vertex of Δ. This implies $A^j \Delta \mathrel{\bar{\prec}} \Delta$, since A is linear. Thus $A^j T \mathrel{\bar{\prec}} T$. ■

Thus "trapping slabs" allow us to determine whether a matrix is Pisot or not. Unfortunately, they also require that we have a priori knowledge of the direction of E^U so that we can create a properly aligned "slab". Ultimately, we will use a similar procedure of "trapping cones" in an effort to approximate the direction of E^U. The following lemma demonstrates how we can use the Perron–Frobenius theorem to determine the containment of E^U in a "trapping cone". This will turn out to be a crucial idea in proving later theorems.

LEMMA 2. Suppose C is a cone with interior in the positive orthant and $\exists\, k$ such that $A^k C \prec C$. Then $E^U \prec C$, where E^U is the unstable eigenspace of A.

PROOF. Suppose not. Suppose that $E^U \nprec C$. Since C is an open cone, there exists a line L in C such that $L \notin E^S$. Since $A \ge 0$ is primitive and $L \ge 0$, by the Perron–Frobenius Theorem [5, p.125] the direction of $A^n L$ approaches that of E^U as $n \to \infty$. Since $E^U \nprec C$ and $A^k C \prec C$, $E^U \nsubseteqq A^k C$, then $\exists\, m$ such that $\forall n \ge m, A^n L \nsubseteqq A^k C$. However, $A^k C \prec C$ implies $\forall n, A^{nk} C \prec C$ and so $A^{(n+1)k} L \in A^{(n+1)k} C \prec A^k C$, which is a contradiction to our supposition. ■

3. "Trapping Cone" Method.

Consider the simplex of $n-1$ dimension

$$S_0 := \left\{ \sum_{i=1}^{n} x_i e_i : \sum_{i=1}^{n} x_i = 1, x_i \ge 0, i = 1,\ldots,n \right\}.$$

Since the vertices of S_0 are the ends of the standard basis vectors in $\mathbb{R}^n$, we will refer to it as the "standard" simplex (**Fig. 4**). Now let us define our "trapping slab" by

$$T_0 := \{p + tv_1 : p \in S_0, t \in \Re\}.$$

Notice that T_0 is an infinite cylinder with cross section S_0 and it contains E^U in its interior by our hypothesis (H) on A.

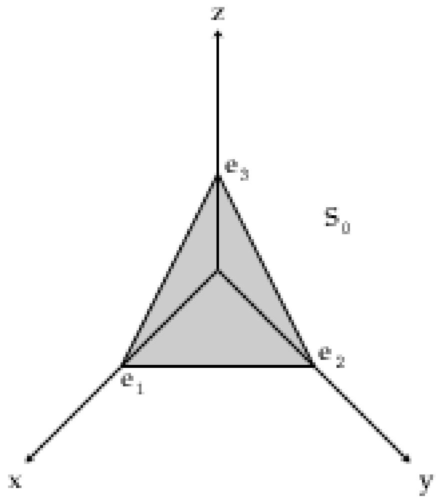


Fig. 4 – Standard simplex for n=3. Notice that it is (n-1) = 2 dimensional surface (plane).

Define

$$C_i(j) = cone(A^j e_i - e_1, \ldots, A^j e_i - e_n) := \left\{ \sum_{k=1}^{n} x_k \left(A^j e_i - e_k \right) : x_k \geq 0 \right\}.$$

LEMMA 3. Then A is Pisot iff $\exists\, j$ such that $\forall i \leq n, E^U \prec C_i(j)$.

PROOF. A is Pisot iff $\exists\, j$ such that $A^j T_0 \precneqq T_0$ by Lemma 1. Since T_0 is a cylinder with edges $L_i = e_i + E^U$ $(i \leq n)$ and A is a linear transformation, $\forall j,\ A^j T_0 \precneqq T_0$ iff $\forall i \leq n, A^j(e_i + E^U) \precneqq T_0$. The latter condition further reduces to $\exists\, j$ such that $\forall i \leq n, A^j e_i + E^U \precneqq T_0$ by virtue of E^U being an eigenspace. Since each $A^j e_i + E^U$ is an infinite line parallel to E^U, strict containment in T_0 is equivalent to passing through the interior of any cross-section of T_0. Let us then consider S_0.

In order for any $A^j e_i + E^U$ to pass through the interior of S_0, this line must be strictly contained in the "fan" of lines connecting $A^j e_i$ to the vertices of S_0 (**Fig. 5**). For simplicity's sake, let us translate these "fans" to the origin, so that we can dispense with discussing a line parallel to E^U and instead consider E^U itself (**Fig. 6**). Upon translation, we find that the condition of "fan" containment is equivalent to $\forall i \le n, E^U \prec C_i(j)$. ■

REMARK 3. When A is Pisot, as j tends to infinity the cones $C_i(j)$ containing E^U become very narrow and close upon E^U because $A^j e_i$ escapes to infinity in the direction of E^U (refer to **Fig. 2,8**).

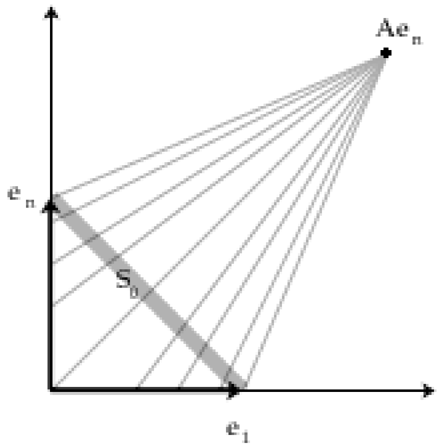


Fig. 5 – *n*th "fan" of lines where n=2.

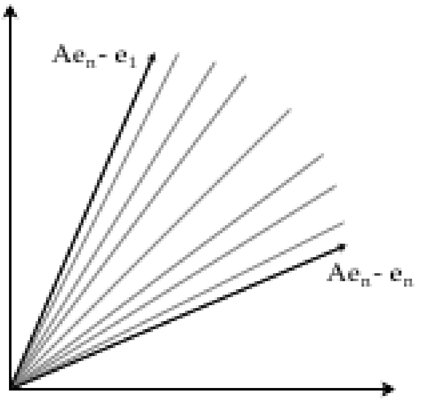


Fig. 6 – Previous figure's "fan" of lines translated to the origin. This cone is $C_n(1)$ where n=2.

Finally, in our main theorem, we can actually rid ourselves of the need for a priori knowledge of E^U.

THEOREM 1. A is Pisot iff $\exists\, j,k$ such that $\forall i \le n, A^k C_i(j) \prec C_i(j)$.

PROOF. ($\Rightarrow$) Suppose A is Pisot. Then E^S is contractive and $\exists\, j$ such that $\forall i \le n, E^U \prec C_i(j)$ by Lemma 3. In view of Remark 3, $C_i(j)$ can be found in the positive orthant and we can use Lemma 0 to find $k > 0$ such that $\forall i \le n, A^k C_i(j) \prec C_i(j)$.

($\Leftarrow$) Suppose A is not Pisot. Then $\forall j, \exists\, i \le n$ such that $E^U \nprec C_i(j)$ by Lemma 3. As before, we may require that $C_i(j)$ is in the positive orthant. Then $\forall k, A^k C_i(j) \nprec C_i(j)$ by Lemma 2, which contradicts our hypothesis. ■

4. Generalized "Pisot" conditions on polynomials.

Since the purpose of this paper is ultimately the discovery of conditions for polynomials and not matrices we will refine our selection of A to an arbitrary polynomial's companion matrix. The restrictions we now place on our "arbitrary" polynomial are required in order for A to fulfill the conditions required by the "trapping cones" method outlined earlier.

We will consider polynomials of the form

$$p(x) = x^n - b_n x^{n-1} - \ldots - b_2 x - b_1$$

where $\forall i > 1, b_i$ are nonnegative integers and b_1 is a positive integer. The condition $b_1 > 0$ is so that $p(x)$'s companion matrix, A_0, is nonsingular. We will exclude polynomials of the form $x^n - 1$, as they are obviously not Pisot. It is easily verified that A_0 is nonnegative and its associated digraph (**Fig. 7**) is strongly connected. Then A_0 is irreducible but not necessarily aperiodic [5, p.3].

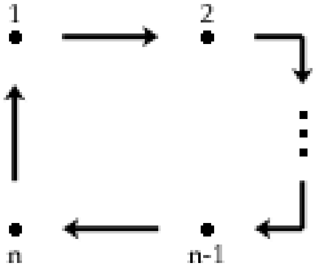


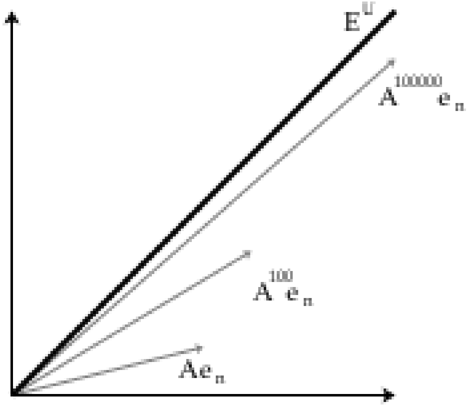
Fig. 7 – The digraph for A_0 where $b_1 > 0$ and $\forall i > 1, b_i = 0$ is strongly connected. The digraph for the general A_0 contains this one and is thus also strongly connected.

Since periodicity of A_0 implies that no eigenvalue of A_0 is simple, if A_0 is periodic then it is not Pisot. Periodicity of A_0 is easy to discern, as follows. Define $P = \{n - i : b_i > 0, 1 < i \leq n\}$. Then by looking at the associated digraph, we find that A_0 is aperiodic iff $\gcd\{P\} = 1$. Thus ruling out the cases where this is not so, we can conclude that A_0 is nonnegative, irreducible, aperiodic, and thus primitive [5, p.125]. Then by the Perron-Frobenius Theorem for primitive matrices [5, p.125]:

a) $\exists$ simple eigenvalue $\lambda_1 \in \mathbb{R}^+$ such that $\forall i > 1, \lambda_1 > |\lambda_i|$
b) $\exists$ corresponding eigenvector $v_1 > 0$
c) $\forall v \geq 0, \lim_{k \to \infty} A_0^{\,k} v \times v_1 = 0$

Since $\left|\prod_{k=1}^{n}\lambda_k\right| = b_1 \geq 1$, then $\exists\, i \leq n \ni |\lambda_i| \geq 1$. Then by a), $\lambda_1 \geq 1$. Suppose $\lambda_1 = 1$. Then $0 = p(\lambda_1) = p(1) = 1 - \sum_{k=1}^{n} b_k \Rightarrow 1 = \sum_{k=1}^{n} b_k$. Given our assumptions on the form of $p(x)$, this only occurs when $p(x) = x^n - 1$ which we have excluded. Thus $\lambda_1 > 1$. This combined with b) satisfies hypothesis (H) required of A in the previous discussion. A major consequence of c) is that $\exists\, k, \forall v \geq 0,\ A_0{}^k v > 0$ by linearity of A_0; that is to say, that some power of A_0 maps the positive orthant strictly inside its interior. Furthermore, we see that applying sufficiently large iterates of A_0 to any vector in the positive orthant will yield improved approximations to E^U (**Fig. 8**). This implies that our "trapping cone" approximation of E^U will improve for higher iterates of A_0, suggesting that it will be more likely to reveal new families of Pisot polynomials by considering those higher iterates.

Before we begin our use of the "trapping cone" method outlined earlier, it is important to notice that our restrictions on A_0 make its behavior significantly easier to predict (**Fig. 9**) and the cone containment conditions easier to verify.



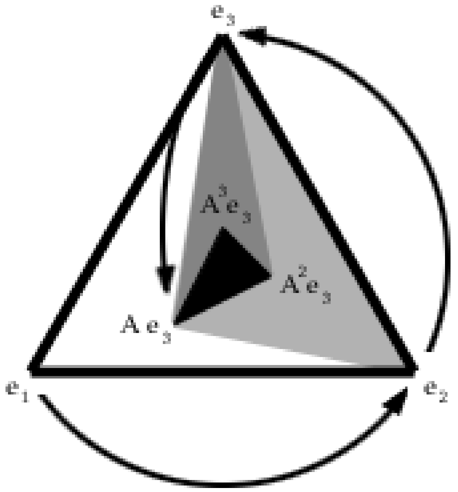


Fig. 8 – Sufficiently large iterates of A_0 to a vector in the positive orthant approach E^U.
Fig. 9 – One of the nicer properties of A_0 is that $\forall i < n,\ A_0 e_i = e_{i+1}$. Figure 9 shows an instance where this might greatly simplify the verification of a "trapping" slab.

We begin with a simplification of the idea of verifying cone containment by standardizing cones. Let $i, j \in \mathbb{N}$ such that $j + i > n$. Then we will define $D_{i,j}$ as the change of coordinate matrix that standardizes $C_i(j)$, taking $A_0{}^j e_i - e_k \rightarrow e_k, \forall k \leq n$.

LEMMA 4. $D_{i,j}$ is invertible

PROOF. If $D_{i,j}$ is a change of coordinate matrix between two bases of $\mathbb{R}^n$ then it is invertible. Thus all that is needed is to prove that $\{A_0{}^j e_i - e_k : \forall\, k \le n\}$ is a linearly independent set and therefore a basis for $\mathbb{R}^n$.

CLAIM. $\{A_0{}^j e_i - e_k : \forall\, k \le n\}$ is a linearly independent set.

PROOF. Suppose $\exists\, c_i$ such that $c_1(A_0{}^j e_i - e_1) + \ldots + c_n(A_0{}^j e_i - e_n) = 0$. Then $A_0{}^j e_i \sum_{k=1}^{n} c_k = [c_1, \ldots, c_n]^T$. Define $[A_0{}^j e_i]_k = a_k$. Then

$$(\lozenge) \qquad \forall\, k, a_k \sum_{m=1}^{n} c_m = c_k .$$

Adding these equations together, we get that

$$\sum_{m=1}^{n} a_m \sum_{m=1}^{n} c_m = \sum_{m=1}^{n} c_m$$

$$\sum_{m=1}^{n} c_m \left(\sum_{m=1}^{n} a_m - 1 \right) = 0$$

$$\sum_{m=1}^{n} a_m = 1 \quad or \quad \sum_{m=1}^{n} c_m = 0 .$$

Now, $\sum_{m=1}^{n} a_m = 1$ together with being nonnegative integers implies $\exists k \le n$ such that $A_0{}^j e_i = e_k$. Since $\forall k \le n,\ A_0{}^{i-k} e_k = e_i$, this implies that $A_0{}^{j+i-k} e_k = e_k$. Notice that $j + i - k > 0$. This contradicts the primitivity of A_0 which implies that $A_0{}^{j+i-k} e_k > 0$. Thus $\sum_{k=1}^{n} c_k = 0$. Then via (◊), $\forall i \le n,\ c_i = 0$ and the set is linearly independent. ■

■

The result which follows will ultimately simplify the conditions required by the previously stated Theorem 1 so much that we call its new form Theorem 2.

THEOREM 2. A_0 is Pisot if any of the following conditions hold:

a) $\exists\, j,k$ such that $\forall i \le n,\ j+i > n \Rightarrow D_{i,j}A_0^{\,k}(D_{i,j})^{-1} \ge 0$ and is primitive.

b) $\exists\, i,k$ such that $\forall j,\ i \le j < i+n \Rightarrow D_{n,j}A_0^{\,k}(D_{n,j})^{-1} \ge 0$ and is primitive.

PROOF. a) Suppose $\exists\, j,k$ such that $\forall i \le n,\ j+i > n \Rightarrow D_{i,j}A_0^{\,k}(D_{i,j})^{-1} \ge 0$ and is primitive.

CLAIM 1. $A_0^{\,j}T_0 \subseteq T_0$

PROOF. $A_0^{\,j}T_0 \subseteq T_0$ is equivalent to $\forall i \le n,\ A_0^{\,j}e_i \in T_0$ by the proof of Lemma 3. Fix $i \le n$. For $j+i \le n$, $A_0^{\,j}e_i = e_{j+i} \in T_0$. For $j+i > n$, we consider $D_{i,j}A_0^{\,k}(D_{i,j})^{-1} \ge 0$. Define the set of points in the positive orthant as O_1. Then $D_{i,j}A_0^{\,kl}(D_{i,j})^{-1} \ge 0$ is equivalent to $D_{i,j}A_0^{\,k}(D_{i,j})^{-1}O_1 \subset O_1$. Since $C_i(j) = (D_{i,j})^{-1}O_1$, this is equivalent to $D_{i,j}A_0^{\,k}C_i(j) \subset O_1$. Applying $(D_{i,j})^{-1}$ to both sides, $A_0^{\,k}C_i(j) \subset C_i(j)$. Then $E^U \subset C_i(j)$ by an argument similar to that found in the proof of Lemma 2. Then $A_0^{\,j}e_i \in T_0$ by the proof of Lemma 3. ■

CLAIM 2. $\exists\, l$ such that $A_0^{\,jl}T_0 \precsim T_0$.

PROOF. $A_0^{\,lj}T_0 \precsim T_0$ is equivalent to $A_0^{\,lj}e_i \precsim T_0$ by the proof of Lemma 3. Fix $i \le n$ such that $j+i > n$. Then the primitivity of $D_{i,j}A_0^{\,k}(D_{i,j})^{-1}$ implies $\exists\, l$ such that $D_{i,j}A_0^{\,kl}(D_{i,j})^{-1} > 0$. Taking the same steps as above, we find that $D_{i,j}A_0^{\,kl}(D_{i,j})^{-1} > 0$ is equivalent to $A_0^{\,kl}C_i(j) \prec C_i(j)$. Then $E^U \prec C_i(j)$ by Lemma 2. Then $A_0^{\,j}e_i \precsim T_0$ by the proof of Lemma 3. Then Claim 1 implies $A_0^{\,jl}e_i \precsim T_0$.

Fix $i \le n$ such that $j+i \le n$. $\exists\, l_i$ such that $n \ge (l_i - 1)j + i > n - j$ and thus we have

$A_0^{(l_i-1)j}e_i = e_{(l_i-1)j+i}$. Since $((l_i-1)j+i)+j>n$, by the primitivity argument above, $A_0^{jl_i}e_i \mathrel{\overline{\prec}} T_0$. Choose $l=\max\{l_i\}$. Then Claim 1 implies $A_0^{jl}e_i \mathrel{\overline{\prec}} T_0$. ■

By combining Claim 2 and Lemma 1, we conclude that A_0 is Pisot. ■

b) By Lemma 1 and the proof of Lemma 3, to prove that A_0 is Pisot it is sufficient to show that $\exists l$ such that $\forall i \le n,\ A_0^{l}e_i \mathrel{\overline{\prec}} T_0$. Suppose $\exists\, i,k$ such that $\forall j,\ i \le j < i+n \Rightarrow D_{n,j}A_0^{k}(D_{n,j})^{-1} \ge 0$ and is primitive. Then $\exists\, m$ such that $\forall j,\ i \le j < i+n, \Rightarrow A_0^{jm}e_n \mathrel{\overline{\prec}} T_0$, by the proof of Claim 2.
Since $\left\{A_0^{im}e_n,\dots,A_0^{(i+n-1)m}e_n\right\} = \left\{A_0^{(i+n-1)m}e_1,\dots,A_0^{(i+n-1)m}e_n\right\}$, then $\forall i \le n,\ A_0^{(i+n-1)m}e_i \mathrel{\overline{\prec}} T_0$. ■

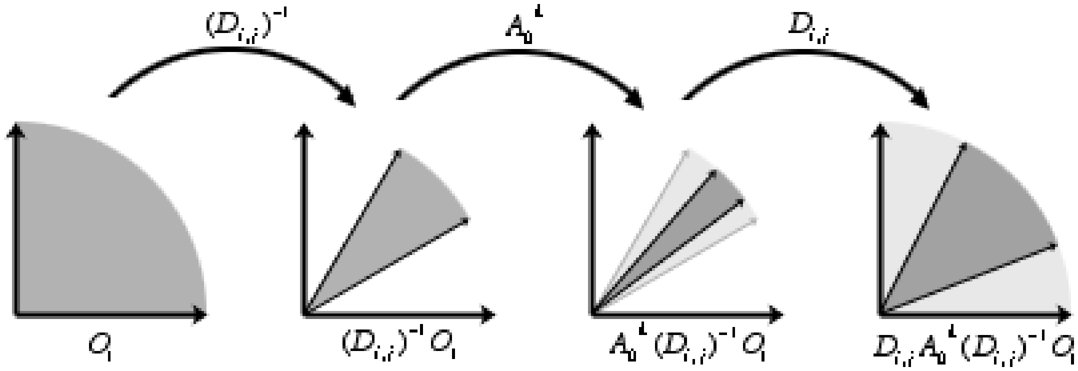


Fig. 10 – Checking for $E^U \prec C_i(j)$ via $D_{i,j}A_0^{kl}(D_{i,j})^{-1} > 0$. Notice that $(D_{i,j})^{-1}O_1 = C_i(j)$.

5. Application to the result of Brauer and Hurwitz.

Let us now apply the simplest form of this method (part a) with $j=1, k=1$) to arbitrary A_0. Then the relevant matrices are

$$A_0 = \begin{vmatrix} 0 & \cdots & \cdots & 0 & b_1 \\ 1 & \ddots & & \vdots & b_2 \\ 0 & \ddots & \ddots & \vdots & b_3 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & b_n \end{vmatrix} \text{ and } D_{n,1}^{-1} = \begin{vmatrix} b_1-1 & b_1 & b_1 & \cdots & b_1 \\ b_2 & b_2-1 & b_2 & \cdots & b_2 \\ b_3 & b_3 & b_3-1 & \cdots & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & b_n & b_n & \cdots & b_n-1 \end{vmatrix}.$$

Since it is rather tedious and messy to calculate $D_{n,1}$, and hence $D_{n,1}A_0(D_{n,1})^{-1}$, using matrix notation, we will introduce the following tensor notation.

Let $a,k\in\mathbb{R}^n$. Define $k\otimes a$ as the linear mapping on $\mathbb{R}^n$ defined by $x\mapsto\langle k|x\rangle a$, and define R as the linear mapping on $\mathbb{R}^n$ defined by $R(x_1,x_2,...,x_n)^T=(0,x_1,...,x_{n-1})^T$. Here $\langle x|y\rangle$ stands for the ordinary dot product on $\mathbb{R}^n$. It will be important in the proof of Rule 3 that we are familiar with the form of R^T, so notice that expressed in matrix notation

$$R=\begin{vmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & \ddots & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{vmatrix} \text{ and } R^T=\begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{vmatrix}.$$

This implies that R^T is defined by $R^T(x_1,...,x_{n-1},x_n)^T=(x_2,...,x_n,0)^T$. We will now establish some rules for this notation.

RULE 1. $\forall a,b,k,l\in\mathbb{R}^n$, $(k\otimes a)\circ(l\otimes b)=\langle k|b\rangle(l\otimes a)$

PROOF. Let $x\in\mathbb{R}^n$. Then

$$\begin{aligned}(k\otimes a)\circ(l\otimes b)(x)&=(k\otimes a)(\langle l|x\rangle b)\\&=\langle l|x\rangle(k\otimes a)(b)\\&=\langle l|x\rangle\langle k|b\rangle a\\&=\langle k|b\rangle(l\otimes a)(x)\end{aligned}$$

■

RULE 2. $\forall a,k\in\mathbb{R}^n$, $R\circ(k\otimes a)=k\otimes Ra$

PROOF. Let $x\in\mathbb{R}^n$. Then

$$\begin{aligned}R\circ(k\otimes a)(x)&=R\langle k|x\rangle a\\&=\langle k|x\rangle Ra\\&=(k\otimes Ra)(x)\end{aligned}$$

■

RULE 3. $\forall a,k \in \mathbb{R}^n$, $(k \otimes a) \circ R = \left(R^T k\right) \otimes a$

PROOF. Let $x \in \mathbb{R}^n$. Then

$$\begin{aligned}(k \otimes a) \circ R(x) &= \langle k | Rx \rangle a \\ &= (k_2 x_1 + k_3 x_2 + \ldots + k_n x_{n-1})a \\ &= \langle R^T k | x \rangle a \\ &= (R^T k \otimes a)(x)\end{aligned}$$

■

The following notational equivalencies are easily verified and are stated without proof. Below, I_n is the $n \times n$ identity matrix.

FACT 1. Define $b = (b_1, b_2, \ldots, b_n)^T$. Then $A_0 = R + (e_n \otimes b)$.

FACT 2. Define $e = (1,1,\ldots,1)^T$. Then ${D_{n,1}}^{-1} = (e \otimes b) - I_n$.

FACT 3. $D_{n,1} = \left(\langle e | b \rangle - 1\right)^{-1} (e \otimes b) - I_n$.

Now we can compute $D_{n,1} A_0 (D_{n,1})^{-1}$ in a much more efficient and tidy manner.

CLAIM 1. $D_{n,1}A_0(D_{n,1})^{-1} = R + e \otimes (b - Rb)$

PROOF.

$$\begin{aligned}
D_{n,1}A_0(D_{n,1})^{-1} &= \Big(\big(\langle e | b \rangle - 1 \big)^{-1} (e \otimes b) - I_n \Big) (R + (e_n \otimes b)) \big((e \otimes b) - I_n \big) \\
&= \Big(\big(\langle e | b \rangle - 1 \big)^{-1} (e \otimes b) - I_n \Big) \Big((e \otimes Rb) - R + \langle e_n | b \rangle (e \otimes b) - (e_n \otimes b) \Big) \\
&= \Big(\big(\langle e | b \rangle - 1 \big)^{-1} (e \otimes b) - I_n \Big) \big((e \otimes Rb) - R + (b_n e - e_n) \otimes b \big) \\
&= \big(\langle e | b \rangle - 1 \big)^{-1} (e \otimes b)(e \otimes Rb) - \big(\langle e | b \rangle - 1 \big)^{-1} (e \otimes b) R + \\
&\quad \big(\langle e | b \rangle - 1 \big)^{-1} (e \otimes b) \big((b_n e - e_n) \otimes b \big) - (e \otimes Rb) + R - (b_n e - e_n) \otimes b \\
&= \big(\langle e | b \rangle - 1 \big)^{-1} \langle e | Rb \rangle (e \otimes b) - \big(\langle e | b \rangle - 1 \big)^{-1} (R^T e \otimes b) + \\
&\quad \big(\langle e | b \rangle - 1 \big)^{-1} \langle e | b \rangle \big((b_n e - e_n) \otimes b \big) - (e \otimes Rb) + R - (b_n e - e_n) \otimes b \\
&= \Big(\big(\langle e | b \rangle - 1 \big)^{-1} \big(\langle e | Rb \rangle e - R^T e + (b_n e - e_n) \big) \Big) \otimes b - (e \otimes Rb) + R \\
&= (e \otimes b) - (e \otimes Rb) + R \\
&= R + e \otimes (b - Rb)
\end{aligned}$$

■

Translating back into matrix notation,

$$\begin{aligned}
D_{n,1}A_0(D_{n,1})^{-1} &= R + e \otimes (b - Rb) \\
(\diamond) \qquad &= R + e \otimes (b_1, b_2 - b_1, \ldots, b_n - b_{n-1})^T \\
&= \left| \begin{array}{ccccc}
b_1 & \cdots & \cdots & \cdots & b_1 \\
b_2 - b_1 + 1 & b_2 - b_1 & \cdots & \cdots & b_2 - b_1 \\
b_3 - b_2 & b_3 - b_2 + 1 & b_3 - b_2 & \cdots & b_3 - b_2 \\
\vdots & & \ddots & & \vdots \\
b_n - b_{n-1} & \cdots & b_n - b_{n-1} & b_n - b_{n-1} + 1 & b_n - b_{n-1}
\end{array} \right|
\end{aligned}$$

Notice that $D_{n,1}A_0(D_{n,1})^{-1} \geq 0$ precisely when $1 \leq b_1 \leq b_2 \leq \ldots \leq b_n$.

CLAIM 2. $D_{n,1}A_0(D_{n,1})^{-1} \geq 0 \Rightarrow D_{n,1}A_0(D_{n,1})^{-1}$ is primitive.

PROOF. Suppose $D_{n,1}A_0(D_{n,1})^{-1} \geq 0$. Then $1 \leq b_1 \leq b_2 \leq \ldots \leq b_n$. Let $i < n$. Then the associated digraph (**Fig. 11**) is strongly connected and thus irreducible. Since there exists a direct path from vertex 1 to vertex 1, the greatest common divisor of all possible path lengths is 1, and thus $D_{n,1}A_0(D_{n,1})^{-1}$ is primitive. ■

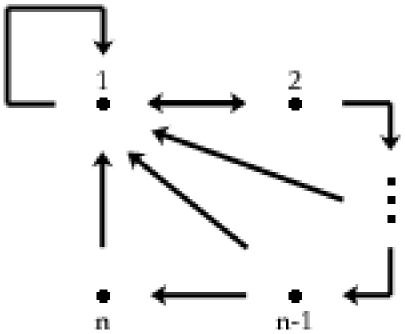


Fig. 11 – The digraph for $D_{n,1}A_0(D_{n,1})^{-1}$ where $b_1 = b_2 = ... = b_n$ is strongly connected and primitive. The digraph for $D_{n,1}A_0(D_{n,1})^{-1}$, where $1 \le b_1 \le b_2 \le ... \le b_n$, will contain at least the paths shown.

Thus we find that the simplest form of Theorem 2 actually provides an alternative geometric proof of the following theorem by Brauer and Hurwitz [1].

THEOREM 3 (B & H). Suppose $p(x) = x^n - b_n x^{n-1} - \ldots - b_2 x - b_1$ with $\forall i \le n$, $b_i \in \mathbb{Z}$. Then $0 < b_1 \le b_2 \le ... \le b_n \Rightarrow p(x)$ is Pisot.

PROOF. We combine Theorem 2, part a), for $j=1, k=1$ with (⋄) and Claim 2. ■

6. Further Results and Possibilities for Continued Research.

In addition to the $j=1, k=1$ case displayed above, I calculated the $k=2$ and $k=3$ cases. Interestingly, contrary to our expectations, incrementing k in Theorem 2 part a) for $j=1$ does not appear to yield further classes of Pisot numbers. This seems counterintuitive to the idea that rotation of the cones by low iterates of A_0 may allow for A_0 to be Pisot despite not displaying cone containment initially.

Continued research could follow two paths:

1. Attempt to explain why further iterations of A_0 for the $j=1$ case do not appear to yield more Pisot families, if that is in fact the case. Using the tensor notation provided in this paper, it may be possible to determine an explicit form for $D_{n,1}A_0{}^n(D_{n,1})^{-1}$.

2. Follow the same method outlined in this paper to calculate the conditions implied by $D_{n,j}A_0{}^k(D_{n,j})^{-1} \geq 0$ and primitive for $j \geq 1$.

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