MATH 3242 **Exam 2** **Name:** ____________________

Instructions: Make sure your handwriting is legible. Show all work for credit.
Restrictions: This exam is closed notes/book. You may not use calculators.

1. (25 points) Find a basis for, and the dimension of, the subspace of P_3 consisting of the cubic polynomials $p(x)$ such that $p(1) = 0$ and $p(-1) = 0$. You may use whatever method you prefer, but make sure that you provide sufficient justification as to why this set forms a basis.

Solution: Let this subspace be called V. Then

$$
\begin{aligned}
V &= \{p(x) \in P_3 : p(1) = 0 \text{ and } p(-1) = 0\} \\
&= \left\{a_0 + a_1x + a_2x^2 + a_3x^3 : a_0 + a_1(1) + a_2(1)^2 + a_3(1)^3 = 0 \text{ and } a_0 + a_1(-1) + a_2(-1)^2 + a_3(-1)^3 = 0\right\} \\
&= \left\{a_0 + a_1x + a_2x^2 + a_3x^3 : a_0 + a_1 + a_2 + a_3 = 0 \text{ and } a_0 - a_1 + a_2 - a_3 = 0\right\} \\
&= \left\{a_0 + a_1x + a_2x^2 + a_3x^3 : a_0 = -a_1 - a_2 - a_3 \text{ and } a_0 - a_1 + a_2 - a_3 = 0\right\} \\
&= \left\{(-a_1 - a_2 - a_3) + a_1x + a_2x^2 + a_3x^3 : (-a_1 - a_2 - a_3) - a_1 + a_2 - a_3 = 0\right\} \\
&= \left\{(-a_1 - a_2 - a_3) + a_1x + a_2x^2 + a_3x^3 : -2a_1 - 2a_3 = 0\right\} \\
&= \left\{(-a_1 - a_2 - a_3) + a_1x + a_2x^2 + a_3x^3 : a_1 = -a_3\right\} \\
&= \left\{-a_2 - a_3x + a_2x^2 + a_3x^3 : a_2, a_3 \in \mathbb{R}\right\} \\
&= \left\{a_2(x^2 - 1) + a_3(x^3 - x) : a_2, a_3 \in \mathbb{R}\right\} \\
&= \text{Span}(x^2 - 1, x^3 - x)
\end{aligned}
$$

Thus a natural candidate for a basis is $\langle x^2 - 1, x^3 - x\rangle$. This obviously spans V so we need only test for linear independence. Since we have

$$
\begin{aligned}
&\alpha(x^2 - 1) + \beta(x^3 - x) = 0 \\
&-\alpha - \beta x + \alpha x^2 + \beta x^3 = 0 \quad \Longrightarrow \quad \alpha = \beta = 0
\end{aligned}
$$

the set is linearly independent and hence $\langle x^2 - 1, x^3 - x\rangle$ forms a basis for V. There are two vectors in this basis, so $\dim(V) = 2$.

2. (25 points) Prove that the map $h : M_{2\times 2} \longrightarrow M_{1\times 3}$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a-b & c-d & a+d \end{pmatrix}$$

is a homomorphism. Then find its nullspace and nullity. Finally, use the nullity to determine the homomorphism's rank without actually constructing its rangespace.

Solution: Let $\alpha, \beta \in \mathbb{R}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in M_{2\times 2}$. Then

$$\begin{aligned} h\left(\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \beta \begin{pmatrix} p & q \\ r & s \end{pmatrix}\right) &= h\left(\begin{pmatrix} \alpha a + \beta p & \alpha b + \beta q \\ \alpha c + \beta r & \alpha d + \beta s \end{pmatrix}\right) \\ &= \begin{pmatrix} (\alpha a + \beta p) - (\alpha b + \beta q) & (\alpha c + \beta r) - (\alpha d + \beta s) & (\alpha a + \beta p) + (\alpha d + \beta s) \end{pmatrix} \\ &= \alpha \begin{pmatrix} a-b & c-d & a+d \end{pmatrix} + \beta \begin{pmatrix} p-q & r-s & p+s \end{pmatrix} \\ &= \alpha \cdot h\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \beta \cdot h\left(\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right) \end{aligned}$$

Thus h is a homomorphism. Since we have

$$\begin{aligned} \mathcal{N}(h) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : h\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} a-b & c-d & a+d \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a = b, c = d, \text{ and } d = -a \right\} \\ &= \left\{ \begin{pmatrix} a & a \\ -a & -a \end{pmatrix} : a \in \mathbb{R} \right\} \\ &= \text{Span}\left(\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\right) \end{aligned}$$

Since the ordered set $\left\langle \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \right\rangle$ spans the nullspace and is obviously linearly independent, it forms a basis for $\mathcal{N}(h)$. Since it contains a single nonzero element, nullity$(h) = 1$. Moreover, we can apply the rank/nullity theorem to get rank$(h) = \dim(M_{2\times 2})-$ nullity$(h) = 4 - 1 = 3$.

3. (25 points) Using the natural bases for P_2 and P_3, find the matrix representations of the homomorphisms

$$h : P_2 \longrightarrow P_3 \text{ defined by } h(a_0 + a_1x + a_2x^2) = 5 + a_0x + \tfrac{a_1}{2}x^2 + \tfrac{a_2}{3}x^3, \text{ and}$$
$$g : P_3 \longrightarrow P_2 \text{ and } g(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2.$$

Then, **using these two matrices**, determine the matrix representation of $g \circ h$.
Hint: If you've had calculus then the result here shouldn't surprise you.

Solution: Let $B = \langle 1, x, x^2 \rangle$ and $C = \langle 1, x, x^2, x^3 \rangle$. Applying h to the elements of B we get

$$h(1) = x \qquad h(x) = \frac{1}{2}x^2 \qquad h(x^2) = \frac{1}{3}x^3$$
$$\downarrow \qquad \downarrow \qquad \downarrow$$
$$\text{Rep}_C\left(h(1)\right) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}_C \qquad \text{Rep}_C\left(h(x)\right) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}_C \qquad \text{Rep}_C\left(h(x^2)\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{pmatrix}_C$$

Similarly, applying g to the elements of C we get

$$g(1) = 0 \qquad g(x) = 1 \qquad g(x^2) = 2x \qquad g(x^3) = 3x^2$$
$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$
$$\text{Rep}_B\left(g(1)\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_B \quad \text{Rep}_B\left(g(x)\right) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_B \quad \text{Rep}_B\left(g(x^2)\right) = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}_B \quad \text{Rep}_B\left(g(x^3)\right) = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}_B$$

Thus we obtain the following matrix representation for our homomorphisms:

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \qquad G = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Finally, we can find the matrix representation of $g \circ h$, call it Q, by simply calculating the product GH.

$$Q = GH = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. (25 points) Determine whether A is invertible and, if so, find its inverse. Verify that your answer is correct by multiplying it with A.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$$

Solution: $\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} \longrightarrow \\ \rho_2 - 2\rho_1 \\ \rho_3 - \rho_1 \end{array} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right) \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \rho_3 + 2\rho_2 \end{array}$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right) \begin{array}{c} \longrightarrow \\ \longrightarrow \\ -\rho_3 \end{array} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \begin{array}{c} \rho_1 - 3\rho_3 \\ \rho_2 + 3\rho_3 \\ \longrightarrow \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \begin{array}{c} \rho_1 - 2\rho_2 \\ \longrightarrow \\ \longrightarrow \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

Thus A is invertible and $A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$.

We can verify this easily by checking that either product (i.e. AB or BA) gives us the identity matrix.

$$\begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix} = \begin{pmatrix} -40+32+9 & -80+80 & -120+48+72 \\ 13-10-3 & 26-25 & 39-15-24 \\ 5-4-1 & 10-10 & 15-6-8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Extra Credit

5. (10 points) Prove that if B is invertible then B^T is invertible and $\left(B^T\right)^{-1} = \left(B^{-1}\right)^T$.

Solution: Let B be invertible. Then B^{-1} exists and

$$B^T \left(B^{-1}\right)^T = \left(B^{-1}B\right)^T = I^T = I, \text{ and}$$
$$\left(B^{-1}\right)^T B^T = \left(BB^{-1}\right)^T = I^T = I.$$

Thus B^T is invertible and $\left(B^T\right)^{-1} = \left(B^{-1}\right)^T$.