MATH 3242 **Exam 1** **Name:** ____________________

Instructions: Make sure your handwriting is legible. Show all work for credit.
Restrictions: This exam is closed notes/book. You may not use calculators.

1. (40 points) Solve the system by **Gauss-Jordan** Elimination. Express the solution set in vector form. Finally, identify a particular solution of the system and the solution set of the associated homogeneous system.

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\ 5x_3 + 10x_4 + 15x_6 &= 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 6 \end{aligned}$$

Solution:

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right) \begin{array}{c} \longrightarrow \\ \rho_2 - 2\rho_1 \\ \longrightarrow \\ \rho_4 - 2\rho_1 \end{array} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right) \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \rho_3 + 5\rho_2 \\ \rho_4 + 4\rho_2 \end{array}$$

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right) \begin{array}{c} \longrightarrow \\ -1 \cdot \rho_2 \\ \frac{1}{6} \cdot \rho_4 \leftrightarrow \rho_3 \\ \longrightarrow \end{array} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{c} \longrightarrow \\ \rho_2 - 3\rho_3 \\ \longrightarrow \\ \longrightarrow \end{array}$$

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{c} \rho_1 + 2\rho_2 \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} \left(\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Rewritten in terms of the free variables, this yields solutions of the form $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -3x_2 - 4x_4 - 2x_5 \\ x_2 \\ -2x_4 \\ x_4 \\ x_5 \\ \frac{1}{3} \end{pmatrix}$.

Thus the solution set of this system is $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{pmatrix} + r \cdot \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} : r, s, t \in \mathbb{R} \right\}$.

The parameterless vector in this set is a particular solution of the system. The remaining parameterized vectors form the solution set of the associated homogeneous system.

2. (15 points) For which values of k will the following system have no solutions? Infinitely many solutions? A unique solution?

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (k^2 - 14)z &= k + 2 \end{aligned}$$

Solution: $\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & k^2-14 & k+2 \end{array}\right) \xrightarrow[\rho_3 - 4\rho_1]{\rho_2 - 3\rho_1} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & k^2-2 & k-14 \end{array}\right) \xrightarrow[\rho_3 - \rho_2]{}$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & k^2-16 & k-4 \end{array}\right)$$

This system has no solutions when $k^2 - 16 = 0$ and $k - 4 \neq 0$, in other words, when $k = -4$.
The system has infinitely many solutions when $k^2 - 16 = 0$ and $k - 4 = 0$, in other words, when $k = 4$.
Finally, the system has a unique solution whenever $k^2 - 16 \neq 0$, in other words, when $k \neq \pm 4$.

3. (15 points) Describe the plane (i.e. write it as a parameterized set of vectors) in $\mathbb{R}^3$ passing through the points $(1, 0, 1)$, $(-2, 11, 4)$ and $(4, 2, -3)$. Does it contain the origin?

Solution: Recall that points can be represented as sets of the form $\{\vec{u}\}$, lines as sets of the form $\{\vec{u} + s \cdot \vec{v} : s \in \mathbb{R}\}$, and planes as sets of the form $\{\vec{u} + s \cdot \vec{v} + t \cdot \vec{w} : s, t \in \mathbb{R}\}$ where $\vec{u}, \vec{v}, \vec{w}$ are arbitrary vectors. We want to find a set of this last form which contains all three given points. There are several ways we can proceed. Perhaps the easiest lies in realizing that a plane is defined by any two distinct lines running through it. Let's check that the lines going from $(1, 0, 1)$ to $(-2, 11, 4)$ and from $(1, 0, 1)$ to $(4, 2, -3)$ are distinct. This is easily seen by noticing that the corresponding vectors (which indicate direction),

$$\vec{v} = \begin{pmatrix} -2-1 \\ 11-0 \\ 4-1 \end{pmatrix} = \begin{pmatrix} -3 \\ 11 \\ 3 \end{pmatrix} \text{ and } \vec{w} = \begin{pmatrix} 4-1 \\ 2-0 \\ -3-1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix},$$

are not constant multiples of each other. Moreover, these lines are easy to construct: simply go to a point on the line (we'll choose the point both lines have in common for convenience) and then move any distance in the direction of that line. Thus these lines can be represented by

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + s \cdot \begin{pmatrix} -3 \\ 11 \\ 3 \end{pmatrix} : s \in \mathbb{R} \right\} \text{ and } \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \cdot \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} : t \in \mathbb{R} \right\}$$

Now we simply combine the sets to get a plane containing both lines (and hence all three points):

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + s \cdot \begin{pmatrix} -3 \\ 11 \\ 3 \end{pmatrix} + t \cdot \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} : s, t \in \mathbb{R} \right\}$$

4. (30 points) Prove or disprove that the set $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a+b+c+d=0\}$ forms a vector space with the usual matrix operations.

Solution: Let $V = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a+b+c+d=0\right\}$ with the usual matrix operations.

Since $V \subset M_{2\times 2}$, a vector space under these operations, V is a vector space if and only if it is a subspace of $M_{2\times 2}$. Thus we need only check that V is closed under addition and scalar multiplication:

AC: Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in V$. Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$.
Since this is a 2×2 matrix with real entries such that

$$(a+e)+(b+f)+(c+g)+(d+h) = (a+b+c+d)+(e+f+g+h) = 0+0 = 0,$$

it is an element of V.

MC: Let $\alpha \in \mathbb{R}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V$. Then $\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$.
Since this is a 2×2 matrix with real entries such that

$$(\alpha a)+(\alpha b)+(\alpha c)+(\alpha d) = \alpha(a+b+c+d) = \alpha(0) = 0,$$

it is an element of V.

Thus V is a subspace of $M_{2\times 2}$ and hence it is a vector space.

Extra Credit

5. (10 points) Find a set to span the subspace $\left\{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} : 3a-b=0 \text{ and } c+2d=0\right\}$ of $\mathbb{R}^4$. (Hint: Parameterize.)

Solution:

$$\left\{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} : b=3a \text{ and } c=-2d\right\} = \left\{\begin{pmatrix} a \\ 3a \\ -2d \\ d \end{pmatrix} : a,d \in \mathbb{R}\right\} = \left\{a \cdot \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \end{pmatrix} + d \cdot \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix} : a,d \in \mathbb{R}\right\}$$

$$= \text{Span}\left(\begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}\right)$$