MATH 3221 **Midterm Exam (take-home)**

Instructions: Write each solution in claim-proof form, even if the solution is short. Make sure your handwriting is legible and that your proofs **use complete sentences**. Provide enough detail so that it is clear to me that you understand why each step of your proof is correct. You must turn in this take-home portion of the exam before you can begin the in-class portion.

Restrictions: The only references you may use are your textbook and your notes from class. You may not consult with anybody. Dishonest conduct will result in a zero for this exam.

1. (25 points) (a) Calculate $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{n}{(n+1)!}$ for a few small values of n.

   (b) Make a conjecture about a formula for this expression.

   (c) Prove your conjecture by mathematical induction.

Solution:

(a)
$$\begin{aligned}\frac{1}{2!} &= \frac{1}{2}\\ \frac{1}{2!}+\frac{2}{3!} &= \frac{5}{6}\\ \frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!} &= \frac{23}{24}\end{aligned}$$

(b) Conjecture: $\forall n \geq 1, \quad \sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}$.

(c) Proof: Let P(n) be the statement that $\sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}$.

Basis Step: Let $n = 1$. Since $\frac{1}{2!} = \frac{1}{2} = 1 - \frac{1}{2!}$, P(1) is true.

Inductive Step: Let $n \geq 1$. Suppose that P(n) is true.

$$\begin{aligned}\text{Then } \sum_{k=1}^{n+1} \frac{k}{(k+1)!} &= \left(\sum_{k=1}^{n} \frac{k}{(k+1)!}\right) + \frac{n+1}{(n+2)!}\\ &= \left(1 - \frac{1}{(n+1)!}\right) + \frac{n+1}{(n+2)!}\\ &= 1 + \frac{-(n+2)+(n+1)}{(n+2)!}\\ &= 1 - \frac{1}{(n+2)!}\end{aligned}$$

Thus P($n+1$) is true.

Therefore P(n) is true for all $n \geq 1$ by the Principle of Mathematical Induction.

2. (25 points) If a is any arbitrary integer, then $6 \mid a(a^2+11)$.

Solution:

Claim: $\forall a \in \mathbb{Z}$, $\exists k \in \mathbb{Z}$ such that $a(a^2+11) = 6k$.

Proof: Let $a \in \mathbb{Z}$. By the division algorithm, there exist $q \in \mathbb{Z}$ and $0 \le r < 6$ st $a = 6q + r$.

Case 1: If $a = 6q$, then $a(a^2+11) = (6q)\left((6q)^2+11\right) = 6\left(q(36q^2+11)\right)$.

Case 2: If $a = 6q+1$, then
$$\begin{aligned} a(a^2+11) &= (6q+1)\left((6q+1)^2+11\right) \\ &= (6q+1)(36q^2+12q+12) \\ &= 6\left((6q+1)(6q^2+2q+2)\right). \end{aligned}$$

Case 3: If $a = 6q+2$, then
$$\begin{aligned} a(a^2+11) &= (6q+2)\left((6q+2)^2+11\right) \\ &= (6q+2)(36q^2+24q+15) \\ &= 6\left((3q+1)(12q^2+8q+5)\right). \end{aligned}$$

Case 4: If $a = 6q+3$, then
$$\begin{aligned} a(a^2+11) &= (6q+3)\left((6q+3)^2+11\right) \\ &= (6q+3)(36q^2+36q+20) \\ &= 6\left((2q+1)(18q^2+18q+10)\right). \end{aligned}$$

Case 5: If $a = 6q+4$, then
$$\begin{aligned} a(a^2+11) &= (6q+4)\left((6q+4)^2+11\right) \\ &= (6q+4)(36q^2+48q+27) \\ &= 6\left((3q+2)(12q^2+16q+9)\right). \end{aligned}$$

Case 6: If $a = 6q+5$, then
$$\begin{aligned} a(a^2+11) &= (6q+5)\left((6q+5)^2+11\right) \\ &= (6q+5)(36q^2+60q+36) \\ &= 6\left((6q+5)(6q^2+10q+6)\right). \end{aligned}$$

In every case, $\exists k \in \mathbb{Z}$ such that $a(a^2+11) = 6k$.

3. (25 points) If $\gcd(a, b) = 1$, then $\gcd(ac, b) = \gcd(c, b)$.

Solution:

Claim: $\forall a, b, c \in \mathbb{Z}$, if $\gcd(a, b) = 1$, then $\gcd(ac, b) = \gcd(c, b)$.

Proof: Let $a, b, c \in \mathbb{Z}$ with $\gcd(a, b) = 1$ and let $d = \gcd(c, b)$. We want to show that $\gcd(ac, b) = d$. Since $\gcd(a, b) = 1$ and $\gcd(c, b) = d$, there are $x, y, m, n \in \mathbb{Z}$ such that $ax + by = 1$ and $cm + bn = d$. Then $d = (cm + bn)(ax + by) = ac(mx) + b(cmy + anx + bny)$. Since d can be expressed as a linear combination of ac and b, $d \geq \gcd(ac, b)$. On the other hand, since $d \mid c$ and $d \mid b$, we have $d \mid ac$ and $d \mid b$. Hence d is a common divisor of ac and b, implying that $d \leq \gcd(ac, b)$. Therefore $d = \gcd(ac, b)$.

4. (25 points) If $a > 0$ and $\sqrt[n]{a} \in \mathbb{Q}$, then $\sqrt[n]{a} \in \mathbb{Z}$.

Solution:

Claim: $\forall a, n \in \mathbb{N}$, if $\sqrt[n]{a} \in \mathbb{Q}$, then $\sqrt[n]{a} \in \mathbb{Z}$.

Proof: Let $a, n \in \mathbb{N}$ with $\sqrt[n]{a} \in \mathbb{Q}$. Then $\exists p, q \in \mathbb{Z}$ such that $\gcd(p, q) = 1$ and $\sqrt[n]{a} = \frac{p}{q}$. The latter implies that $aq^n = p^n$. From this we can conclude that $q \mid p^n$. Applying Euclid's Lemma, we get that $q \mid p$. However, since $\gcd(p, q) = 1$, this implies that $q = 1$. Thus $\sqrt[n]{a} = \frac{p}{q} = p \in \mathbb{Z}$.