MATH 3221 HW 3

Instructions: Write each solution in claim-proof form, even if the solution is short. Make sure your handwriting is legible and that your proofs **use complete sentences**. Provide enough detail so that it is clear to me that you understand why each step of your proof is correct. I will not accept late assignments, so it is in your best interests to submit your homework on time *even if it is incomplete*.

1. (5 points) Prove that the only prime of the form $n^3 - 1$ is 7. **Hint:** First find a factorization of $n^3 - 1$.

Solution:

Claim: $\forall n \geq 1$, if $n^3 - 1$ is prime, then $n = 7$.

Proof: Let $n \geq 1$ such that $n^3 - 1$ is prime. Since $n^3 - 1 = (n-1)(n^2+n+1)$, $n^3 - 1$ being prime implies that either $n - 1 = 1$ or $n^2 + n + 1 = 1$. Clearly $n^2 + n + 1 \geq 3$, so it must be the case that $n - 1 = 1$. Thus $n = 2$ and $n^3 - 1 = 7$.

2. (5 points) Prove that the only prime p for which $3p + 1$ is a perfect square is $p = 5$.

Solution:

Claim: $\forall$ prime p, if $\exists k \in \mathbb{Z}$ such that $3p + 1 = k^2$, then $p = 5$.

Proof: Let p be a prime such that $\exists k \in \mathbb{Z}$ with $3p + 1 = k^2$. Since $3(2) + 1 = 7$ and $3(3) + 1 = 10$ are not perfect squares we can assume that $p > 3$. Notice that $3p = k^2 - 1 = (k-1)(k+1)$. p and 3 are prime with $p > 3$ so either $k - 1 = 1$ and $k + 1 = 3p$, or $k - 1 = 3$ and $k + 1 = p$.

Case 1: Suppose $k - 1 = 1$ and $k + 1 = 3p$. Then $k = 2$ and $p = 1$, which contradicts $p > 3$.

Case 2: Suppose $k - 1 = 3$ and $k + 1 = p$. Then $k = 4$ and $p = 5$.

Therefore $p = 5$.

3. (5 points) Prove that each integer $n > 11$ can be written as the sum of two composite numbers.
Hint: If n is even, so that $n = 2k$, then consider $n - 6 = 2(k-3)$; for odd n, consider $n - 9$.

Solution:

Claim: $\forall n > 11$, $\exists$ composite $j, m \in \mathbb{N}$ such that $n = j + m$.

Proof: Let $n > 11$. By the division algorithm, there exists $k \in \mathbb{Z}$ such that $n = 2k$ or $n = 2k + 1$. Notice that $k \geq 6$ since $n \geq 12$.

Case 1: Suppose $n = 2k$. Notice that $n = (n-6) + 6$. Since $6 = 2 \cdot 3$ and $n - 6 = 2k - 6 = 2(k-3)$, where $k - 3 \geq 3$, these are both composite numbers.

Case 1: Suppose $n = 2k + 1$. Notice that $n = (n-9) + 9$. Since $9 = 3 \cdot 3$ and $n - 9 = (2k+1) - 9 = 2(k-4)$, where $k - 4 \geq 2$, these are both composite.

Thus in every case, we can find composite $j, m \in \mathbb{N}$ such that $n = j + m$.

4. (5 points) Prove that if $a \equiv b \pmod{n}$ and $m \mid n$, then $a \equiv b \pmod{m}$.

Solution:

Claim: $\forall a, b \in \mathbb{Z}, \forall m, n \in \mathbb{N}$, if $a \equiv b \pmod{n}$ and $m \mid n$, then $a \equiv b \pmod{m}$.

Proof: Let $a, b \in \mathbb{Z}$ and $m, n \in \mathbb{N}$ such that $a \equiv b \pmod{n}$ and $m \mid n$. Since $m \mid n$ and $n \mid a - b$, it follows that $m \mid a - b$. Thus $a \equiv b \pmod{m}$.

5. (5 points) Give a counterexample to the following statement: $a^3 \equiv b^3 \pmod{n} \Rightarrow a \equiv b \pmod{n}$.

Solution:

Claim: $\exists a, b \in \mathbb{Z}, \exists n \in \mathbb{N}$ such that $a^3 \equiv b^3 \pmod{n}$ and $a \not\equiv b \pmod{n}$.

Proof: Choose $a = 1, b = 2, n = 7$. Then $1^3 \equiv 2^3 \pmod{7}$ and $1 \not\equiv 2 \pmod{7}$.

6. (5 points) Use congruences to find the remainder when 41^{65} is divided by 7.

Solution:

Since $41^{65} \equiv (-1)^{65} \equiv -1 \equiv 6 \pmod{7}$, the remainder when 41^{65} is divided by 7 is 6.

7. (5 points) Prove that $111^{333} + 333^{111}$ is divisible by 7.

Solution:

Claim: $111^{333} + 333^{111} \equiv 0 \pmod{7}$.

Proof: $111^{333} + 333^{111} \equiv (-1)^{333} + (-3)^{111} \equiv -1 + (-27)^{37} \equiv -1 + (1)^{37} \equiv 0 \pmod{7}$.

8. (5 points) For $n \geq 1$, use congruences to show that $43 \mid 6^{n+2} + 7^{2n+1}$.

Solution:

Claim: $\forall n \in \mathbb{N}, \quad 6^{n+2} + 7^{2n+1} \equiv 0 \pmod{43}$.

Proof: Let $n \in \mathbb{N}$. Then $6^{n+2} + 7^{2n+1} = 36 \cdot 6^n + 7 \cdot 49^n \equiv 36 \cdot 6^n + 7 \cdot (6)^n \equiv 43 \cdot 6^n \equiv 0 \pmod{43}$.