MATH 3221 **HW 2**

Instructions: Write each solution in claim-proof form, even if the solution is short. Make sure your handwriting is legible and that your proofs **use complete sentences**. Provide enough detail so that it is clear to me that you understand why each step of your proof is correct. I will not accept late assignments, so it is in your best interests to submit your homework on time *even if it is incomplete.*

1. (5 points) Show that any integer of the form $6k+5$ is also of the form $3j+2$, but not conversely.

> **Solution:**
>
> Claim: $\forall n \geq 1$, if $\exists k \in \mathbb{Z}$ such that $n = 6k+5$, then $\exists j \in \mathbb{Z}$ such that $n = 3j+2$.
>
> Proof: Let $n \geq 1$ and suppose that $\exists k \in \mathbb{Z}$ such that $n = 6k+5$.
>
> Then $n = 6k+5 = 6k+(3+2) = 3(2k+1)+2 = 3j+2$, where $j = 2k+1$.
>
> Claim: $\exists n \geq 1$ and $\exists j \in \mathbb{Z}$ such that $n = 3j+2$ yet $\forall k \in \mathbb{Z}$, $n \neq 6k+5$.
>
> Proof: Choose $n = 2$ so that $2 = 3(0)+2$. Then $2 = 6k+5$ implies that $k = -\frac{1}{2} \notin \mathbb{Z}$.

2. For $n \geq 1$, establish the following:

   (a) (5 points) The integer $n(7n^2+5)$ is of the form $6k$.

> **Solution:**
>
> Claim: $\forall n \geq 1$, $\exists k \in \mathbb{Z}$ such that $n(7n^2+5) = 6k$.
>
> Proof: Let $n \geq 1$. By the division algorithm, there exist $q \in \mathbb{Z}$ and $0 \leq r < 6$ st $n = 6q+r$.
>
> Case 1: If $n = 6q$, then $n(7n^2+5) = (6q)\left(7(6q)^2+5\right) = 6\left(q(252q^2+5)\right)$.
>
> Case 2: If $n = 6q+1$, then $n(7n^2+5) = (6q+1)\left(7(6q+1)^2+5\right) = (6q+1)(252q^2+84q+12) = 6\left((6q+1)(42q^2+14q+2)\right)$.
>
> Case 3: If $n = 6q+2$, then $n(7n^2+5) = (6q+2)\left(7(6q+2)^2+5\right) = (6q+2)(252q^2+168q+33) = 6\left((3q+1)(84q^2+56q+11)\right)$.
>
> Case 4: If $n = 6q+3$, then $n(7n^2+5) = (6q+3)\left(7(6q+3)^2+5\right) = (6q+3)(252q^2+252q+68) = 6\left((2q+1)(126q^2+126q+34)\right)$.

Case 5: If $n = 6q + 4$, then
$$\begin{aligned} n(7n^2+5) &= (6q+4)\big(7(6q+4)^2+5\big) \\ &= (6q+4)(252q^2+336q+117) \\ &= 6\big((3q+2)(84q^2+112q+39)\big). \end{aligned}$$

Case 6: If $n = 6q + 5$, then
$$\begin{aligned} n(7n^2+5) &= (6q+5)\big(7(6q+5)^2+5\big) \\ &= (6q+5)(252q^2+420q+180) \\ &= 6\big((6q+5)(42q^2+70q+30)\big). \end{aligned}$$

In every case, $\exists k \in \mathbb{Z}$ such that $n(7n^2+5) = 6k$.

(b) (5 points) $5 \mid \big(3^{3n+1} + 2^{n+1}\big)$. **Hint:** use induction.

Solution:

Claim: $\forall n \geq 0, 5 \mid \big(3^{3n+1} + 2^{n+1}\big)$.

Proof: Let $\mathrm{P}(n)$ be the statement that $5 \mid \big(3^{3n+1} + 2^{n+1}\big)$.

Basis Step: Let $n = 0$. Since $3^1 + 2^1 = 5 = 5 \cdot 1$, $\mathrm{P}(0)$ is true.

Inductive Step: Let $n \geq 0$. Suppose that $\mathrm{P}(n)$ is true.

That is, suppose $\exists k \in \mathbb{Z}$ such that $3^{3n+1} + 2^{n+1} = 5k$.

Then
$$\begin{aligned} 3^{3(n+1)+1} + 2^{(n+1)+1} &= 3^{3n+4} + 2^{n+2} \\ &= 27 \cdot 3^{3n+1} + 2 \cdot 2^{n+1} \\ &= 27(3^{3n+1} + 2^{n+1}) - 27 \cdot 2^{n+1} + 2 \cdot 2^{n+1} \\ &= 27(5k) - 25 \cdot 2^{n+1} \\ &= 5\big(27k - 5 \cdot 2^{n+1}\big) \\ &= 5m, \text{ where } m = 27k - 5 \cdot 2^{n+1}. \end{aligned}$$

Thus $\mathrm{P}(n+1)$ is true.

Therefore $\mathrm{P}(n)$ is true for all $n \geq 0$ by the Principle of Mathematical Induction.

3. (5 points) Prove that the product of four consecutive integers is 1 less than a perfect square.
Hint: Recall that $k^2 - 1 = (k-1)(k+1)$.

Solution:

Claim: $\forall n \in \mathbb{Z}$, $\exists k \in \mathbb{N}$ such that $n(n+1)(n+2)(n+3) = k^2 - 1$.

Proof: Let $n \in \mathbb{Z}$. Noticing that $n(n+3) = n^2 + 3n$ and $(n+1)(n+2) = n^2 + 3n + 2$, we choose $k = n^2 + 3n + 1$.

Then $k^2 - 1 = (k-1)(k+1) = (n^2+3n)(n^2+3n+2) = n(n+1)(n+2)(n+3)$.

4. Prove the following properties of the greatest common divisor:

(a) (5 points) If $\gcd(a,b) = 1$ and $c \mid a$, then $\gcd(b,c) = 1$.

Solution:

Claim: $\forall a, b, c \in \mathbb{Z}$, if $\gcd(a,b) = 1$ and $c \mid a$, then $\gcd(b,c) = 1$.

Proof: Let $a, b, c \in \mathbb{Z}$ with $\gcd(a,b) = 1$ and $c \mid a$. Then $\exists k, x, y \in \mathbb{Z}$ such that $1 = ax + by$ and $a = kc$. Since $1 = ax + by = (kc)x + by = c(kx) + b(y)$, we have that $\gcd(b,c) = 1$.

(b) (5 points) If $\gcd(a,b) = 1$, then $\gcd(a^2, b^2) = 1$. **Hint:** First show that $\gcd(a, b^2) = 1$.

Solution:

Claim: $\forall a, b \in \mathbb{Z}$, if $\gcd(a,b) = 1$, then $\gcd(a^2, b^2) = 1$.

Proof: Let $a, b \in \mathbb{Z}$ with $\gcd(a,b) = 1$. Then $\exists x, y \in \mathbb{Z}$ such that $1 = ax + by$. Since $1 = (ax+by)^2 = a^2x^2 + 2abxy + b^2y^2 = a(ax^2 + 2bxy) + b^2(y^2)$, we have $\gcd(a, b^2) = 1$. Thus we've shown that $\gcd(a,b) = 1$ implies $\gcd(a, b^2) = 1$. Letting $c = b^2$, this says that $\gcd(c, a) = 1$ implies $\gcd(c, a^2) = 1$. In other words, $\gcd(a^2, b^2) = 1$.

5. Use the Euclidean Algorithm to find $\gcd(a,b)$. Then find integers x and y satisfying $\gcd(a,b) = ax + by$.

(a) (5 points) $a = 119, b = 272$.

Solution: Applying the Euclidean Algorithm,

$$\begin{aligned} 272 &= 2 \cdot 119 + 34 \\ 119 &= 3 \cdot 34 + 17 \\ 34 &= 2 \cdot 17 \qquad \Rightarrow \qquad \gcd(119, 272) = 17 \end{aligned}$$

Then, working our way backwards,

$$\begin{aligned} 17 &= 119 - 3 \cdot 34 \\ &= 119 - 3 \cdot (272 - 2 \cdot 119) \\ &= 7 \cdot 119 + (-3) \cdot 272 \end{aligned}$$

(b) (5 points) $a = 1769, b = 2378$.

Solution: Applying the Euclidean Algorithm,

$$\begin{aligned} 2378 &= 1 \cdot 1769 + 609 \\ 1769 &= 2 \cdot 609 + 551 \\ 609 &= 1 \cdot 551 + 58 \\ 551 &= 9 \cdot 58 + 29 \\ 58 &= 2 \cdot 29 \qquad \Rightarrow \qquad \gcd(1769, 2378) = 29 \end{aligned}$$

Then, working our way backwards,

$$\begin{aligned} 29 &= 551 - 9 \cdot 58 \\ &= 551 - 9 \cdot (609 - 551) \\ &= -9 \cdot 609 + 10 \cdot 551 \\ &= -9 \cdot 609 + 10 \cdot (1769 - 2 \cdot 609) \\ &= 10 \cdot 1769 - 29 \cdot 609 \\ &= 10 \cdot 1769 - 29 \cdot (2378 - 1769) \\ &= 39 \cdot 1769 + (-29) \cdot 2378 \end{aligned}$$