MATH 3221 **HW 1**

Instructions: Write each solution in claim-proof form, even if the solution is short. Make sure your handwriting is legible and that your proofs **use complete sentences**. Provide enough detail so that it is clear to me that you understand why each step of your proof is correct. I will not accept late assignments, so it is in your best interests to submit your homework on time *even if it is incomplete*.

1. For all $n \geq 1$, prove the following by mathematical induction:

   (a) (5 points) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$.

Solution:

Claim: $\forall n \geq 1, \quad \displaystyle\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$.

Proof: Let P(n) be the statement that $\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$.

Basis Step: Let $n = 1$. Since $\displaystyle\sum_{k=1}^{1} k(k+1) = (1)(2) = 2$ and $\dfrac{1(1+1)(1+2)}{3} = 2$, P$(1)$ is true.

Inductive Step: Let $n \geq 1$. Suppose that P(n) is true.

Then
$$\begin{aligned}\sum_{k=1}^{n+1} k(k+1) &= \left[\sum_{k=1}^{n} k(k+1)\right] + (n+1)(n+2) \\ &= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) \\ &= \frac{n(n+1)(n+2) + 3(n+1)(n+2)}{3} \\ &= \frac{(n+1)(n+2)(n+3)}{3} \\ &= \frac{(n+1)((n+1)+1)((n+1)+2)}{3}.\end{aligned}$$

Thus P$(n+1)$ is true.

Therefore P(n) is true for all $n \geq 1$ by the Principle of Mathematical Induction.

(b) (5 points) $a + ar + ar^2 + \cdots + ar^n = \dfrac{a(r^{n+1} - 1)}{r - 1}$ for any $r \neq 1$.

Solution: (Here we rule out the trivial case where $r = 0$ and prove the statement for $n \geq 0$.)

Claim: $\forall a, r \in \mathbb{R}$ with $r \neq 0, r \neq 1, \forall n \geq 0$, $\displaystyle\sum_{k=0}^{n} ar^k = \frac{a(r^{n+1} - 1)}{r - 1}$.

Proof: Let $a, r \in \mathbb{R}$ with $r \neq 0, r \neq 1$. Let $\mathrm{P}(n)$ be the statement that $\sum_{k=0}^{n} ar^k = \frac{a(r^{n+1}-1)}{r-1}$.

Basis Step: Let $n = 0$. Since $\displaystyle\sum_{k=0}^{0} ar^k = a$ and $\dfrac{a(r^{0+1} - 1)}{r - 1} = a$, $\mathrm{P}(0)$ is true.

Inductive Step: Let $n \geq 0$. Suppose that $\mathrm{P}(n)$ is true.

Then
$$\begin{aligned}
\sum_{k=0}^{n+1} ar^k &= \left[\sum_{k=0}^{n} ar^k\right] + ar^{n+1} \\
&= \frac{a(r^{n+1} - 1)}{r - 1} + ar^{n+1} \\
&= \frac{a(r^{n+1} - 1) + ar^{n+1}(r - 1)}{r - 1} \\
&= \frac{a(r^{n+2} - 1)}{r - 1} \\
&= \frac{a(r^{(n+1)+1} - 1)}{r - 1}.
\end{aligned}$$

Thus $\mathrm{P}(n + 1)$ is true.

Therefore $\mathrm{P}(n)$ is true for all $n \geq 0$ by the Principle of Mathematical Induction.

(c) (5 points) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}\le 2-\frac{1}{n}$.

Solution:

Claim: $\forall n \geq 1, \quad \displaystyle\sum_{k=1}^{n}\frac{1}{k^2}\le 2-\frac{1}{n}.$

Proof: Let P(n) be the statement that $\sum_{k=1}^{n}\frac{1}{k^2}\le 2-\frac{1}{n}$.

Basis Step: Let $n=1$. Since $\displaystyle\sum_{k=1}^{1}\frac{1}{k^2}=1$ and $2-\frac{1}{1}=1$, P(1) is true.

Inductive Step: Let $n\geq 1$. Suppose that P(n) is true.

Then
$$\begin{aligned}
\sum_{k=1}^{n+1}\frac{1}{k^2} &= \left[\sum_{k=1}^{n}\frac{1}{k^2}\right]+\frac{1}{(n+1)^2}\\
&\le \left[2-\frac{1}{n}\right]+\frac{1}{(n+1)^2}\\
&= 2+\frac{-(n+1)^2+n}{n(n+1)^2}\\
&= 2-\frac{n^2+n+1}{n(n+1)^2}\\
&= 2-\frac{n^2+n}{n(n+1)^2}-\frac{1}{n(n+1)^2}\\
&= 2-\frac{1}{n+1}-\frac{1}{n(n+1)^2}\\
&\le 2-\frac{1}{n+1}.
\end{aligned}$$

Thus P($n+1$) is true.

Therefore P(n) is true for all $n\geq 1$ by the Principle of Mathematical Induction.

(d) (5 points) $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$.

Solution:

Claim: $\forall n \geq 1, \quad \sum_{k=1}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n}$.

Proof: Let P(n) be the statement that $\sum_{k=1}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n}$.

Basis Step: Let $n = 1$. Since $\sum_{k=1}^{1} \frac{k}{2^k} = \frac{1}{2}$ and $2 - \frac{1+2}{2^1} = \frac{1}{2}$, P(1) is true.

Inductive Step: Let $n \geq 1$. Suppose that P(n) is true.

$$\text{Then} \quad \sum_{k=1}^{n+1} \frac{k}{2^k} = \left[\sum_{k=1}^{n} \frac{k}{2^k}\right] + \frac{n+1}{2^{n+1}}$$

$$= \left[2 - \frac{n+2}{2^n}\right] + \frac{n+1}{2^{n+1}}$$

$$= 2 + \frac{-2(n+2) + (n+1)}{2^{n+1}}$$

$$= 2 - \frac{n+3}{2^{n+1}}$$

$$= 2 - \frac{(n+1)+2}{2^{(n+1)}}.$$

Thus P($n+1$) is true.

Therefore P(n) is true for all $n \geq 1$ by the Principle of Mathematical Induction.

(e) (5 points) $n^2 - n$ is even.

Solution:

Claim: $\forall n \geq 1, \exists k \in \mathbb{Z}$ such that $n^2 - n = 2k$.

Proof: Let P(n) be the statement that $\exists k \in \mathbb{Z}$ such that $n^2 - n = 2k$.

Basis Step: Let $n = 1$. Since $1^2 - 1 = 0 = 2(0)$, P(1) is true.

Inductive Step: Let $n \geq 1$. Suppose that P(n) is true.

That is, suppose there exists $k \in \mathbb{Z}$ such that $n^2 - n = 2k$.

$$\begin{aligned} \text{Then } (n+1)^2 - (n+1) &= (n^2 + 2n + 1) - n - 1 \\ &= n^2 + n \\ &= (n^2 - n) + 2n \\ &= 2k + 2n \\ &= 2(k + n). \end{aligned}$$

Thus P($n+1$) is true.

Therefore P(n) is true for all $n \geq 1$ by the Principle of Mathematical Induction.

2. (5 points) If $2 \leq k \leq n-2$, show that

$$\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k} \qquad n \geq 4$$

Solution:

Claim: $\forall n \geq 4, \forall k \in \mathbb{Z}$ with $2 \leq k \leq n-2$, $\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}$.

Proof: Let $n \geq 4$ and $2 \leq k \leq n-2$.

(Easy way) Then, using Pascal's Rule, we have

$$\begin{aligned}\binom{n}{k} &= \binom{n-1}{k-1} + \binom{n-1}{k} \\ &= \left[\binom{n-2}{k-2} + \binom{n-2}{k-1}\right] + \left[\binom{n-2}{k-1} + \binom{n-2}{k}\right] \\ &= \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}.\end{aligned}$$

(Tedious way) Then we have

$$\begin{aligned}\tbinom{n-2}{k-2} + 2\tbinom{n-2}{k-1} + \tbinom{n-2}{k} &= \tfrac{(n-2)!}{(k-2)![(n-2)-(k-2)]!} + 2\tfrac{(n-2)!}{(k-1)![(n-2)-(k-1)]!} + \tfrac{(n-2)!}{k![(n-2)-k]!} \\ &= \frac{(n-2)!}{(k-2)!(n-k)!} + 2\frac{(n-2)!}{(k-1)!(n-k-1)!} + \frac{(n-2)!}{k!(n-k-2)!} \\ &= \frac{k(k-1)(n-2)! + 2k(n-k)(n-2)! + (n-k)(n-k-1)(n-2)!}{k!(n-k)!} \\ &= \frac{(n-2)!}{k!(n-k)!}\left[k(k-1) + 2k(n-k) + (n-k)(n-k-1)\right] \\ &= \frac{(n-2)!}{k!(n-k)!}\left[k^2 - k + 2kn - 2k^2 + (n^2 - 2kn - n + k^2 + k)\right] \\ &= \frac{(n-2)!}{k!(n-k)!}(n^2 - n) \\ &= \frac{n(n-1)(n-2)!}{k!(n-k)!} \\ &= \frac{n!}{k!(n-k)!} \\ &= \binom{n}{k}.\end{aligned}$$

3. (5 points) Prove the following for $n \geq 1$:

$$\binom{n}{r} < \binom{n}{r+1} \qquad \text{iff} \qquad 0 \leq r < \frac{1}{2}(n-1)$$

Solution:

Claim: $\forall n \geq 1, \binom{n}{r} < \binom{n}{r+1}$ iff $0 \leq r < \frac{1}{2}(n-1)$.

Proof: Let $n \geq 1$. Suppose that $\displaystyle\binom{n}{r} < \binom{n}{r+1}$. Notice that this implies that $0 \leq r \leq n-1$.

Then $$\frac{n!}{r!(n-r)!} < \frac{n!}{(r+1)![n-(r+1)]!}$$

$$\frac{(r+1)!}{r!} < \frac{(n-r)!}{[n-r-1)]!}$$

$$r+1 < n-r$$

$$r < \frac{1}{2}(n-1).$$

Thus $0 \leq r < \dfrac{1}{2}(n-1)$.

Since all of these steps are reversible, $0 \leq r < \frac{1}{2}(n-1)$ implies that $\displaystyle\binom{n}{r} < \binom{n}{r+1}$.

4. (5 points) Establish the inequality $2^n < \binom{2n}{n} < 2^{2n}$, for $n > 1$.

   Hint: Let $x = 2 \cdot 4 \cdot 6 \cdots (2n), y = 1 \cdot 3 \cdot 5 \cdots (2n-1)$, and $z = 1 \cdot 2 \cdot 3 \cdots n$. Now show that $x > y > z$, hence $x^2 > xy > xz$.

Solution:

Claim: $\forall n > 1, \quad 2^n < \binom{2n}{n} < 2^{2n}$.

Proof: Let $n > 1$. Consider the quantities

$$x = 2 \cdot 4 \cdot 6 \cdots (2n) = \prod_{k=1}^{n} 2k,$$

$$y = 1 \cdot 3 \cdot 5 \cdots (2n-1) = \prod_{k=1}^{n} 2k - 1, \text{ and}$$

$$z = 1 \cdot 2 \cdot 3 \cdots n = \prod_{k=1}^{n} k.$$

These are all products of n terms. Notice that for any $k \geq 1$, the kth term of x, $2k$, is strictly larger than the kth term of y, $2k-1$. Similarly, the kth term of y, $2k-1$, is strictly larger than (except at $k = 1$ when they are equal) the kth term of z, k. Together these imply that $z < y < x$. Hence $xz < xy < x^2$. Now recall that

$$x = 2^n \cdot n!, \qquad y = \frac{(2n)!}{2^n \cdot n!}, \qquad \text{and} \qquad z = n!.$$

Then

$$xz < xy < x^2$$

$$2^n \cdot (n!)^2 < \frac{(2n)! \cdot 2^n \cdot n!}{2^n \cdot n!} < 2^{2n}(n!)^2$$

$$2^n < \frac{(2n)!}{(n!)^2} < 2^{2n}$$

$$2^n < \frac{(2n)!}{n!(2n-n)!} < 2^{2n}$$

$$2^n < \binom{2n}{n} < 2^{2n}.$$