MATH 3221 **HW 1** **Due Friday, June 12**

Instructions: Write each solution in claim-proof form, even if the solution is short. Make sure your handwriting is legible and that your proofs **use complete sentences**. Provide enough detail so that it is clear to me that you understand why each step of your proof is correct. I will not accept late assignments, so it is in your best interests to submit your homework on time *even if it is incomplete*.

1. For all $n \geq 1$, prove the following by mathematical induction:

   (a) (5 points) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$.

   (b) (5 points) $a + ar + ar^2 + \cdots + ar^n = \dfrac{a(r^{n+1} - 1)}{r - 1}$ for any $r \neq 1$.

   (c) (5 points) $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots + \dfrac{1}{n^2} \leq 2 - \dfrac{1}{n}$.

   (d) (5 points) $\dfrac{1}{2} + \dfrac{2}{2^2} + \dfrac{3}{2^3} + \cdots + \dfrac{n}{2^n} = 2 - \dfrac{n+2}{2^n}$.

   (e) (5 points) $n^2 - n$ is even.

2. (5 points) If $2 \leq k \leq n-2$, show that

$$\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k} \qquad n \geq 4$$

3. (5 points) Prove the following for $n \geq 1$:

$$\binom{n}{r} < \binom{n}{r+1} \quad \text{iff} \quad 0 \leq r < \frac{1}{2}(n-1)$$

4. (5 points) Establish the inequality $2^n < \binom{2n}{n} < 2^{2n}$, for $n > 1$.

   Hint: Let $x = 2 \cdot 4 \cdot 6 \cdots (2n), y = 1 \cdot 3 \cdot 5 \cdots (2n-1)$, and $z = 1 \cdot 2 \cdot 3 \cdots n$. Now show that $x > y > z$, hence $x^2 > xy > xz$.