8. Important Properties of Logarithmic Functions

Up until now our experience with logarithms has made them seem horribly unwieldy. While the actual calculation of values isn't very pleasant, there do exist a number of powerful properties of logarithms that allow us to simplify our dealings with them. Interestingly enough, all of these properties arise from the fundamental fact that logarithms of numbers are exponents.

I. Overview

DO

The Product Rule:	$\log_a MN = \log_a M + \log_a N$
The Power Rule:	$\log_a M^p = p \log_a M$
The Quotient Rule:	$\log_a \frac{M}{N} = \log_a M - \log_a N$
The Change-of-Base Rule:	$\log_b M = \frac{\log_a M}{\log_a b}$
Inverse Function Property:	$a^{\log_a x} = x$ $\log_a a^x = x$ $\log_a 1 = 0$ $\log_a a = 1$

DON'T

$\log_a MN \neq (\log_a M)(\log_a N)$
$\log_a (M+N) \neq \log_a M + \log_a N$
$\log_a \frac{M}{N} \neq \frac{\log_a M}{\log_a N}$
$(\log_a M)^p \neq p \log_a M$

Having these rules memorized and being able to use them in combination with each other greatly simplifies many calculus problems. Let us look briefly at why these rules are true.

II. Proofs of the Various Rules

The only rules we need to prove from scratch are the Product Rule and the Power Rule.

Proof of the Product Rule: $\log_a M$ and $\log_a N$ are some unknown values – let's call them m and n respectively.

Then $\log_a M = m$ and $\log_a N = n$.
Hence $M = a^m$ and $N = a^n$, so $MN = a^m a^n = a^{m+n}$.
Since $MN = a^{m+n}$, then $\log_a MN = m + n$.
But since $m = \log_a M$ and $n = \log_a N$,

$$\boxed{\log_a MN = \log_a M + \log_a N}$$

Proof of the Power Rule: Again, let $\log_a M = m$.

Then $M = a^m$ and so $M^p = (a^m)^p = a^{mp}$.

Since $M^p = a^{mp}$, then $\log_a M^p = mp$.

But since $m = \log_a M$, $\boxed{\log_a M^p = p \log_a M}$

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Now the other two rules follow straight from these two rules.

Proof of the Quotient Rule:

$$\begin{aligned}\log_a \frac{M}{N} &= \log_a MN^{-1} \\ &= \log_a M + \log_a N^{-1} \\ &= \boxed{\log_a M - \log_a N}\end{aligned}$$

Proof of the Change-of-Base Rule: Let $\log_b M = m$.

Then $M = b^m$ and so $\log_a b^m = \log_a M$

Hence $m \log_a b = \log_a M$, and so

$$m = \frac{\log_a M}{\log_a b}$$

But since $m = \log_b M$, $\boxed{\log_b M = \dfrac{\log_a M}{\log_a b}}$

III. Inverse Function Properties

The last four entries in the table are all direct consequences of the Inverse Function Property. If you can recall:

Inverse Function Property: $\left(f \circ f^{-1}\right)(x) = f^{-1}(f(x)) = x$ for all x in the domain of $f^{-1}(x)$

$\left(f^{-1} \circ f\right)(x) = f\left(f^{-1}(x)\right) = x$ for all x in the domain of $f(x)$

We'll say that $f(x) = a^x$ and $f^{-1}(x) = \log_a x$.

1. Then
$$\begin{aligned}\left(f \circ f^{-1}\right)(x) &= x \\ f\left(f^{-1}(x)\right) &= x \\ f\left(\log_a x\right) &= x \\ a^{(\log_a x)} &= x\end{aligned}$$

2. Similarly
$$\begin{aligned}\left(f^{-1} \circ f\right)(x) &= x \\ f^{-1}\left(f(x)\right) &= x \\ f^{-1}\left(a^x\right) &= x \\ \log_a\left(a^x\right) &= x\end{aligned}$$

The remaining two listed are just special cases of number 2 above:

3. $\log_a 1 = \log_a a^0 = 0$, so $\boxed{\log_a 1 = 0}$.

4. $\log_a a = \log_a a^1 = 1$, so $\boxed{\log_a a = 1}$.

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IV. Examples, Examples, Examples

Examples on the Board:

Express as a sum of logarithms.

1. $\log_2(6)$

2. $\log_6(14x)$

Express as a product.

3. $\log_a x^4$

4. $\ln\sqrt{a}$

Express as a difference of logarithms.

5. $\log_a \frac{76}{13}$

6. $\log_b \frac{3}{w}$

Express in terms of sums and differences of logarithms.

7. $\ln\frac{5a}{4b^2}$

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8. $\log_c \sqrt[3]{\frac{y^3 z^2}{x^4}}$

Express as a single logarithm and, if possible, simplify.

9. $\ln 2x + 3(\ln x - \ln y)$

10. $\frac{2}{3}\left[\ln\left(x^2 - 9\right) - \ln(x+3)\right] + \ln(x+y)$

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Given that $\log_b 2 \approx .693$, $\log_b 3 \approx 1.099$, and $\log_b 5 \approx 1.609$, find each of the following.

11. $\log_b \frac{5}{3}$

12. $\log_b 15b$

Simplify.

13. $\log_q q^{\sqrt{3}}$

14. $e^{\ln x^3}$

15. $\log_b \sqrt{b^3}$