

7. Exponential and Logarithmic Functions

Exponential functions are found everywhere. They are used to model everything from the rate of bacterial growth in biology to the rate of decay of radioactive substances in chemistry (ever heard the terms half-life or carbon dating?).

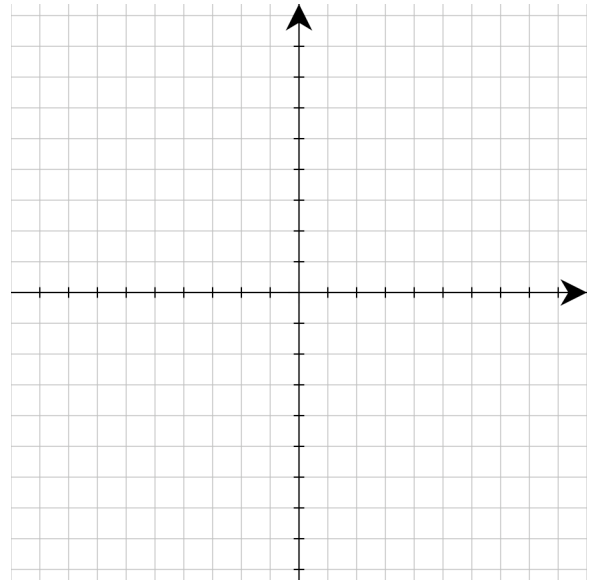
Def: The function $f(x) = a^x$, with $a > 0$ and $a \neq 1$, is called the **exponential function of base a**.

Why do we place restrictions on the value of a ?

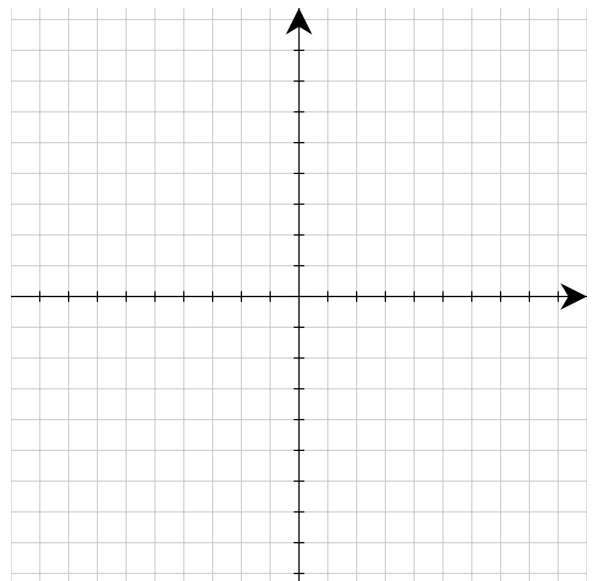
Examples on the board:

Make a table of test points and graph the following functions.

1). $f(x) = 3^x$



2). $f(x) = \frac{1}{3}^x$



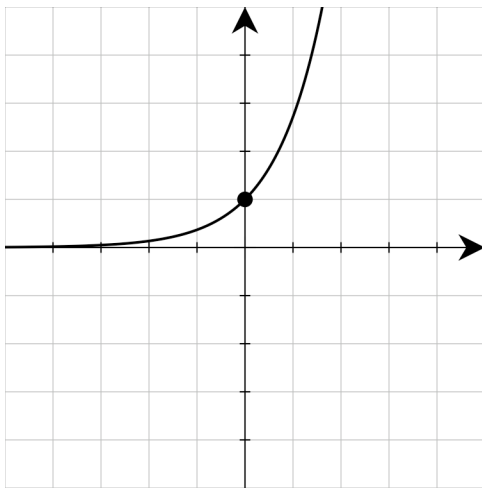
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Remark: While using 0 or 1 for a base is technically fine, this actually yields a constant function. Since these end up being oddities among functions of the form $f(x) = a^x$, we don't consider them to be exponential functions. Using a negative number for a base is not done for a different reason; such a function would be undefined for most numbers since non-integer inputs correspond to taking roots of a negative number.

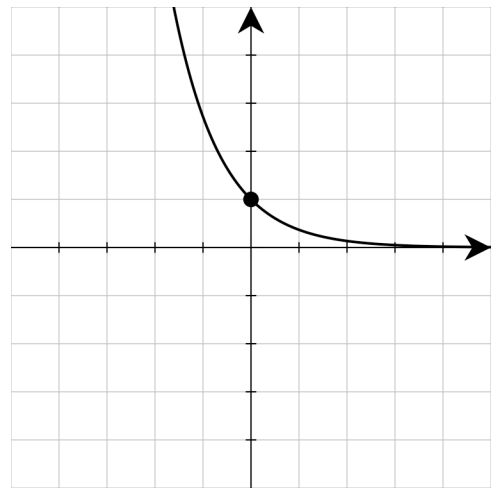
I. Properties of Exponential Functions

The graphs of exponential functions fall into one of two general categories:

If $a > 1$, then the graph of $f(x) = a^x$ is **increasing** like so



If $0 < a < 1$, then the graph of $f(x) = a^x$ is **decreasing** like so



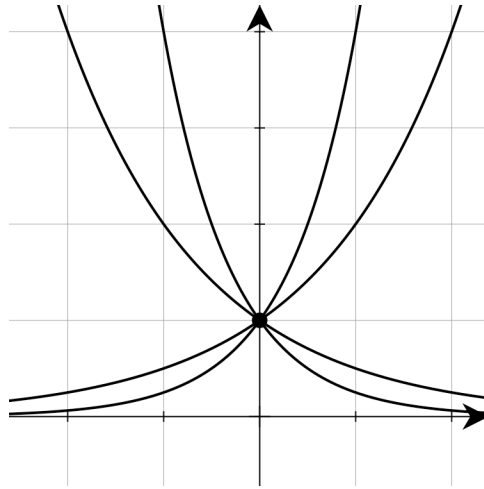
All exponential functions:

1. have a y-intercept at (0,1)
2. have the x-axis as a horizontal asymptote (this limiting behavior is only on one side)
3. are either always increasing or always decreasing (dependant on the base)
4. have domain \mathbb{R}
5. are one-to-one
6. are continuous
7. are smooth

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Examples on the board

Match the following functions with their graphs: $f(x) = 2^x$, $g(x) = 2^{-x}$, $h(x) = 4^x$, $k(x) = 4^{-x}$

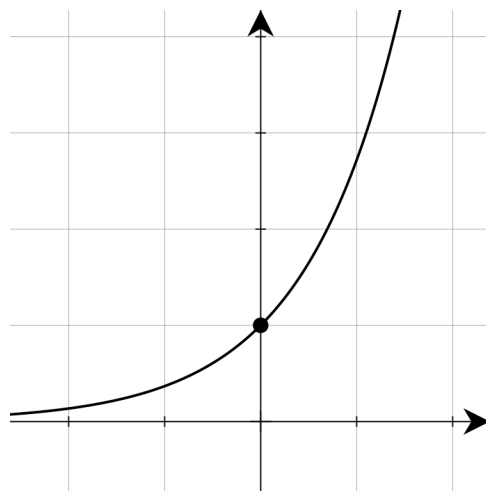


In application, there is one base which shows up more often than any other. As such, it is deemed to be the “natural” exponential function. This number is similar to π , in that it is irrational (an infinitely non-repeating decimal) and it seems to be a crucial quantity in a mind-boggling number of seemingly unrelated applications. The rest of the chapter we will be focusing on the use of this particular number as a base.

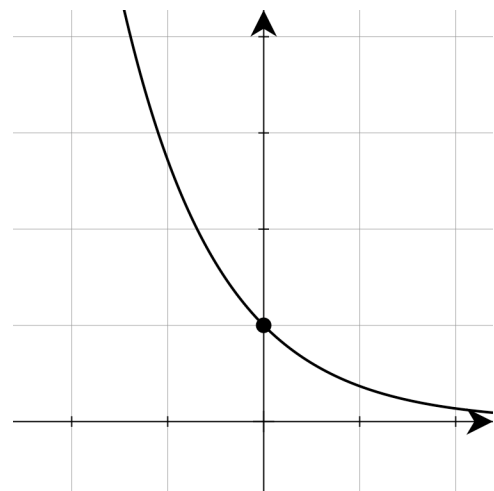
II. The number e

DEF:

e is the irrational number ($e \approx 2.71828\dots$) defined to be the value that the formula $\left(1 + \frac{1}{n}\right)^n$ approaches as n goes to ∞ . Alternatively, it is the horizontal asymptote of the function $f(x) = \left(1 + \frac{1}{x}\right)^x$.



graph of e^x



graph of e^{-x}

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III. Logarithmic Functions

When we begin trying to solve exponential equations, we'll see that there aren't many tools for solving for a variable that's "stuck" in an exponent. In fact, we'll take a new approach – we'll solve these equations using the inverse functions of exponential equations (recall that all exponential functions are one-to-one, so they have inverse functions).

If we try to find the inverse of an exponential function $f(x) = a^x$ in the usual fashion we get:

$$y = a^x$$

at which point, we get stuck. How do we solve for y ? Intuitively, we can see that y is the power to which we have to raise the base a to get x , but we don't want to have to write that down everytime we solve a problem like this. Seeing this, mathematicians decided to create a notation for saying exactly that. So when we get into this situation we write $y = \log_a x$ and it means " y is the power to which a must be raised to get x ". Thus we can continue:

$$y = a^x$$

$$x = \log_a y$$

$$f^{-1}(y) = \log_a y$$

- Note:**
1. $y = \log_a x$ **means** $x = a^y$. It's just a new notation that lets us solve for y .
 2. When we start investigating the properties of logarithmic functions next section, it will be helpful to remember that the logarithm of a number is an **exponent** (it is the exponent in $x = a^y$).

Def: The **logarithmic function of base a** is written $f(x) = \log_a x$, and is the inverse of the exponential function $f(x) = a^x$. However, two bases are used more commonly than any others, so there are special notations for these:

Common logarithms: (base 10) is written without a subscript as $\log x$

Natural logarithms: (base e) is written as $\ln x$

Ex: Calculate $\log_4 16$.

Remember, this means "the power to which we have to raise 4 to get 16."
Thus $\log_4 16 = 2$.

Let's check it. We know that $\log_4 16 = 2$ **means** $16 = 4^2$, which is obviously the case.

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Examples on the board:

Calculate each of the following logarithms.

1. $\log_2 32$

2. $\log_{16} 4$

3. $\log_a 1$ for any base a

4. $\log_a a$ for any base a

Convert each of the following to a logarithmic equation.

5. $5^{-3} = \frac{1}{125}$

6. $e^{-t} = 4000$

Convert each of the following to an exponential equation.

7. $\log x = 14$

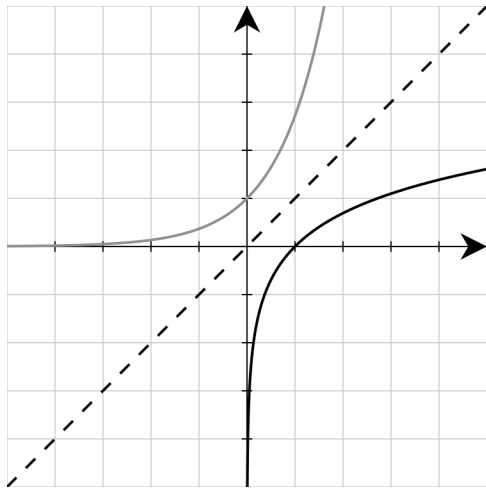
8. $\log_x y = z$

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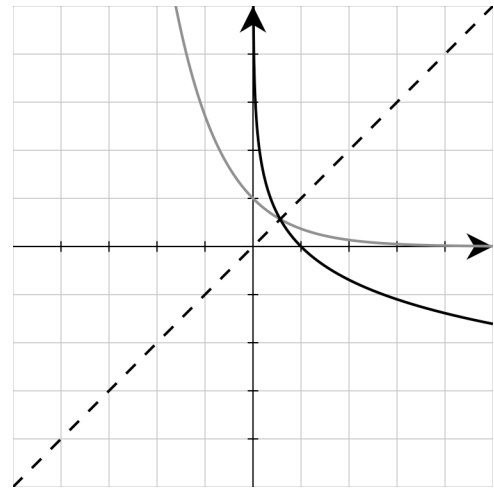
IV. Properties of Logarithmic Functions

Just as with exponential functions, the graphs of logarithmic functions fall into one of two general categories based on the base a .

If $a > 1$, the graph of $f(x) = \log_a x$ is **increasing** like so



If $0 < a < 1$, the graph of $f(x) = \log_a x$ is **decreasing** like so



Logarithmic function are in black, corresponding inverse (exponential) functions are in grey.

All logarithmic functions:

1. have an x-intercept at (1,0)
2. have the y-axis as a vertical asymptote
3. have domain $(0, \infty)$
4. are either always increasing or always decreasing (dependant on the base)
5. are one-to-one
6. are continuous
7. are smooth

Q: Which properties are the same as in exponential functions and which are different? How are the differences related?

A:

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Notice: On your scientific calculator, there should be buttons labeled “log” and “ln”. However, it also means that we need to know how to change other bases to either base 10 or e if we want to calculate logarithms of other bases. We can do this using the following formula.

Change-of-Base Formula: $\log_b M = \frac{\log_a M}{\log_a b}$

Examples on the board:

Write $\log_5 12$ in terms of natural logarithms.