6. Partial Fractions

In algebra, when confronted with rational expressions or equations we find it useful (for simplifying, solving, graphing, etc.) to combine the various separate fractions into one large fraction using the LCD.

For example:

$$
\begin{aligned}
f(x) &= \frac{3}{x+2} - \frac{2}{2x-3} \\
&= \frac{3(2x-3)}{(x+2)(2x-3)} - \frac{2(x+2)}{(2x-3)(x+2)} \\
&= \frac{3(2x-3)-2(x+2)}{(2x-3)(x+2)} \\
&= \frac{6x-9-2x-4}{(2x-3)(x+2)} \\
&= \frac{4x-13}{(2x-3)(x+2)}
\end{aligned}
$$

Now we can more easily find zeros, vertical tangents, holes in the graph, and so forth.

I say this is true "in algebra" because in calculus the reverse is often true. Various calculus procedures are significantly easier if you can express complicated rational expressions, like

$$\frac{4x-13}{(2x-3)(x+2)},$$

in terms of several simpler fractions, like

$$\frac{3}{x+2} - \frac{2}{2x-3}.$$

So we now examine a technique used to do just that – **partial fraction decomposition**. We'll run through the steps with an example problem. Let us first look at a definition that will be important to the procedure.

Def: A quadratic factor (ax^2+bx+c) is **irreducible** if it cannot be factored into linear factors $(ax+b)$ with rational coefficients. In other words, it has no rational zeros.

Ex: Is x^2-x-1 irreducible?

Using the quadratic formula, we see that x^2-x-1 has zeros $\frac{1\pm\sqrt{5}}{2}$. Since it can only have two zeros, so we know that the only linear factors of x^2-x-1 are irrational. Thus it is irreducible.

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Setup: If the degree of the denominator is larger than the degree of the numerator, then you can skip to step 1. If this is not the case, then you need to do polynomial long division (and then you'll do partial fractions on the remainder).

Step 1: Factor the denominator into linear factors $(px+q)^n$ and **irreducible** quadratic factors $\left(ax^2+bx+c\right)^k$. If there are any common factors with the numerator, cancel them out.

Step 2: Now we set our rational function equal to a general partial fraction decomposition of the appropriate form. The appropriate form is easily constructed by doing the following

(a) For each linear factor $(px+q)^n$ in the denominator of the rational function, the partial fraction decomposition should contain the sum of n "mystery" partial fractions of the form

$$\frac{A_1}{px+q}+\frac{A_2}{(px+q)^2}+\cdots+\frac{A_n}{(px+q)^n}$$

(b) Similarly, for each quadratic factor $\left(ax^2+bx+c\right)^k$ in the denominator of the rational function, the partial fraction decomposition should contain the sum of k "mystery" partial fractions of the form

$$\frac{B_1x+C_1}{ax^2+bx+c}+\frac{B_2x+C_2}{\left(ax^2+bx+c\right)^2}+\cdots+\frac{B_kx+C_k}{\left(ax^2+bx+c\right)^k}$$

Step 3: Our only remaining goal is to solve for all these unknown constants (A_i, B_i, C_i). This can be done in a number of ways. Using a combination of these methods, we can solve most partial fractions problems very quickly.

To outline the different methods below, we'll look at an example. Let's perform partial fraction decomposition on

$$\frac{x^2-2}{(x+3)(x-1)^2}$$

Since the degree of the denominator is larger than that of the numerator, we can skip the Setup step. We can also skip Step 1 since the denominator is already factored. Then according to Step 2, we should have

$$\frac{x^2-2}{(x+3)(x-1)^2}=\frac{A}{x+3}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

(I could have used A_1, A_2, A_3 for my constants like I used above, but this is unpleasant. You only need to make sure that each unknown constant is a different letter.)

Method 1 : **Collecting Like Terms** (can always be used, but is the most tedious)

1. Clear all fractions by multiplying both sides of the equation by the LCD (least common denominator). This gives

$$x^2-2=A(x-1)^2+B(x+3)(x-1)+C(x+3)$$

2. Expand the right-hand side and then collect like terms.

$$x^2-2=A(x^2-2x+1)+B(x^2+2x-3)+C(x+3)$$
$$x^2-2=(A+B)x^2+(-2A+2B+C)x+(A-3B+3C)$$

3. The coefficients of like terms on left- and right-hand sides have to be equal. Set up and solve these equations.

$$A + B = 1$$
$$B = 1 - A$$

$$-2A + 2B + C = 0$$
$$-2A + 2(1 - A) + C = 0$$
$$2 - 4A + C = 0$$
$$C = 4A - 2$$

$$A - 3B + 3C = -2$$
$$A - 3(1 - A) + 3(4A - 2) = -2$$
$$A - 3 + 3A + 12A - 6 = -2$$
$$16A = 7$$
$$\boxed{A = \tfrac{7}{16}}$$
$$\boxed{B = \tfrac{9}{16}}$$
$$\boxed{C = -\tfrac{1}{4}}$$

4. Write out the partial decomposition.

$$\frac{x^2 - 2}{(x+3)(x-1)^2} = \frac{\frac{7}{16}}{x+3} + \frac{\frac{9}{16}}{x-1} - \frac{\frac{1}{4}}{(x-1)^2}$$

Method 2 : **Substitution** (can always be used, but works best with distinct linear terms)

1. Clear all fractions by multiplying both sides of the equation by the LCD (least common denominator). This gives

$$x^2 - 2 = A(x-1)^2 + B(x+3)(x-1) + C(x+3)$$

2. This equation is true for all values of x, so we can substitute in values for x and use the resulting equations to solve for our constants. When you have linear terms, you should always try substituting their zeros.

Substituting $x = 1$, we get

$$1^2 - 2 = C(1+3)$$
$$-1 = 4C$$
$$\boxed{C = -\tfrac{1}{4}}$$

Substituting $x = -3$, we get

$$(-3)^2 - 2 = A(-3-1)^2$$
$$7 = 16A$$
$$\boxed{A = \tfrac{7}{16}}$$

We've run out of zeros to substitute, with one constant remaining. At this point you substitute using any other values of x. Substituting $x = 0$, we get

$$0^2 - 2 = \tfrac{7}{16}(0-1)^2 + B(0+3)(0-1) - \tfrac{1}{4}(0+3)$$
$$-2 = \tfrac{7}{16} - 3B - \tfrac{3}{4}$$
$$-\tfrac{27}{16} = -3B$$
$$\boxed{B = \tfrac{9}{16}}$$

3. Write out the partial decomposition.

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Method 3 : **Heaviside Cover-up Method** (easiest, but only gives constant for the highest power of a linear term)

1. The Cover-up method only lets you find the constant for the partial fraction with highest power of a linear term (so here we could find A and C, but not B). To do this you find the zero of that linear term and then plug this zero into the left-hand side, “covering up” that same linear factor.

$$\frac{x^2-2}{(x+3)(x-1)^2}=\frac{A}{x+3}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

So to find A, we’d note that its linear factor $x+3$ has zero $x=-3$.
Then, covering up the factor $x+3$ on the left-hand side, we plug in $x=-3$, to get

$$A=\frac{(-3)^2-2}{(-3-1)^2}=\boxed{\frac{7}{16}}$$

Similarly, C is the constant corresponding to the partial fraction with the largest power of the linear term $x-1$ (which has zero $x=1$). Then, covering up the factor $(x-1)^2$ on the left-hand side, we plug in $x=1$, to get

$$C=\frac{1^2-2}{(1+3)}=\boxed{-\frac{1}{4}}$$

2. Use one of the other two methods to solve for the remaining constants and write out the full decomposition.

Examples on the Board

Decompose into partial fractions.

1. $\dfrac{2x}{(x+1)(x-1)}$

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2. $\dfrac{5x^3+6x^2+5x}{(x^2-1)(x+1)^3}$

3. $\dfrac{11x^2-39x+16}{(x^2+4)(x-8)}$