4. Rational Functions and Equations

Notice that while adding, subtracting, or multiplying polynomials together gives you another polynomial, dividing polynomials by one another need not result in a polynomial. This means we need to add another class of functions to our catalog.

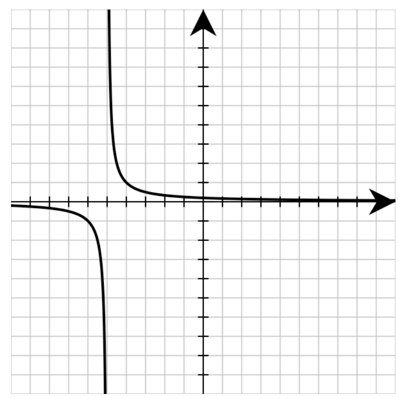
Def: A **rational function** is a function $f(x)$ that is the quotient of two polynomials:

$$f(x) = \frac{p(x)}{q(x)}, \text{ where } p(x) \text{ and } q(x) \text{ are polynomials with } q(x) \neq 0.$$

Recall that the domain of a rational function is all real numbers except where $q(x) = 0$.

Note: If the domain of a rational function does not include all of $\mathbb{R}$, then the graph will not be continuous. Notice the behavior of $f(x) = \frac{1}{x+5}$ near $x = 5$.

x	$f(x)$
-5.1	-10
-5.01	-100
-5.00001	-100000
-4.99999	100000
-4.99	100
-4.9	10

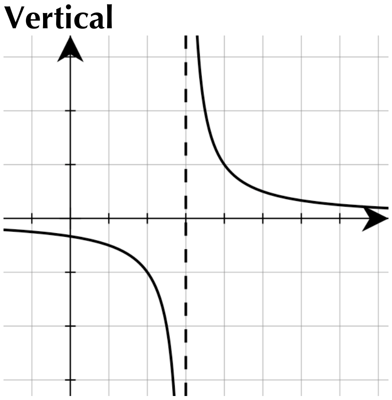


I. Asymptotes

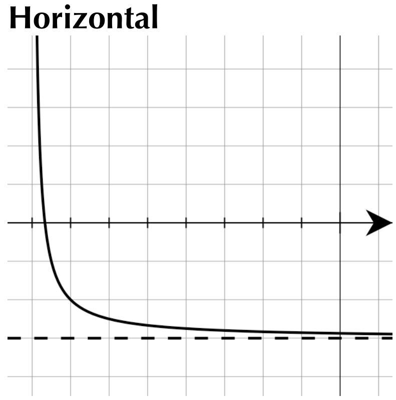
Def (Informal): A function f has an **asymptote** if there exists a straight line with the following property: as one moves along the graph of the function f, the distance between it and the straight line (asymptote) becomes progressively smaller, and can in fact be made as small as desired by moving far enough on the graph of f.

There are three types of asymptotes:

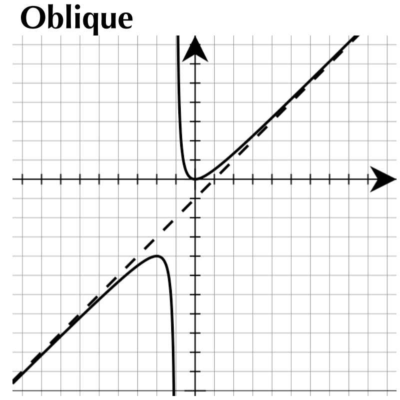
Vertical



Horizontal



Oblique



4. Rational Functions and Equations

Def (Informal):

1. The vertical line $x = c$ is a **vertical asymptote** for the function f if the graph of f goes to either positive or negative infinity as it approaches $x = c$.

   Intuitively, a vertical asymptote acts like an invisible vertical barrier to the graph of the function. A function <u>cannot</u> cross a vertical asymptote.

2. The horizontal line $y = b$ is a **horizontal asymptote** for the function f if the distance between the graph of f and the line $y = b$ gets progressively smaller as x goes to positive **and/or** negative infinity (not necessarily both).

   Intuitively, the limiting behavior of at least one side of the function looks like the line $y = b$. A function <u>can</u> cross a horizontal asymptote.

3. The slanted line $y = mx + b$, $m \neq 0$ is an **oblique asymptote** for the function f if the distance between the graph of f and the line $y = mx + b$ gets progressively smaller as x goes to positive **and/or** negative infinity (not necessarily both).

   Intuitively, the limiting behavior of at least one side of the function looks like the line $y = mx + b$. A function <u>can</u> cross an oblique asymptote.

<u>II. Determining Asymptotes</u>

1. Vertical Asymptotes

A rational function $f(x) = \dfrac{p(x)}{q(x)}$ has a vertical asymptote at $x = c$ if c is a zero of **only** $q(x)$ **and not** $p(x)$. However, if c is a zero of **both**, then the situation is a little more complicated. In this case, $f(x)$ only has a vertical asymptote at $x = c$ if the multiplicity of c is larger for the denominator than the numerator. Otherwise, there is a hole in the graph at $x = c$, but no vertical asymptote.

Example on the board:

1. Find any vertical asymptotes and/or holes of $f(x) = \dfrac{x-2}{x^3 - 4x}$.

4. Rational Functions and Equations

2. Horizontal Asymptotes

A rational function $f(x) = \frac{p(x)}{q(x)}$ has a horizontal asymptote if the degree of $p(x)$ **is less than or equal to** the degree of $q(x)$. If the degree of $p(x)$ is

- **less than** the degree of $q(x)$, then $y = 0$ is the horizontal asymptote.
- **equal to** the degree of $q(x)$, then $y = \frac{a}{b}$ is the horizontal asymptote where a is the leading coefficient of $p(x)$ and b is the leading coefficient of $q(x)$.

Examples on the board:

2. Find the horizontal asymptote of $f(x) = \frac{8x^4 + x - 2}{2x^4 - 10}$

3. Find the horizontal asymptote of $f(x) = \frac{x+6}{4x^3 + 2x^2}$.

3. Oblique Asymptotes

A rational function $f(x) = \frac{p(x)}{q(x)}$ has an oblique asymptote if the degree of $p(x)$ **is exactly 1 greater than** the degree of $q(x)$. The oblique asymptote of $f(x)$ is $y = mx + b$ where $mx + b$ is the quotient resulting from dividing $q(x)$ into $p(x)$.

Example on the board:

4. Find the oblique asymptote of $f(x) = \frac{3x^3 + 2x - 6}{-x^2 + 3x - 2}$.

4. Rational Functions and Equations

Note: A graph can only have one horizontal asymptote or one oblique asymptote, never both. Also remember that an asymptote is **not** a part of the graph of the function; it is an imaginary guideline that we use to help visualize the shape of the graph of the function.

III. Solving Rational Equations

Consider the rational equation $\frac{3}{x+2}+\frac{2}{x}=\frac{4x-4}{x^2-4}$.

Step 1: Factor denominators in order to find the Least Common Denominator (LCD).

$$\frac{3}{x+2}+\frac{2}{x}=\frac{4(x-1)}{(x-2)(x+2)}$$ Thus the LCD $= x(x-2)(x+2)$.

Step 2: Multiply both sides of the equation by the LCD. This will rid us of all fractions.

$$\left[x(x-2)(x+2)\right]\left[\frac{3}{x+2}+\frac{2}{x}\right]=\left[\frac{4(x-1)}{(x-2)(x+2)}\right]\left[x(x-2)(x+2)\right]$$
$$3x(x-2)+2(x-2)(x+2)=4x(x-1)$$

Step 3: Simplify and solve as usual.

$$3x(x-2)+2(x-2)(x+2)=4x(x-1)$$
$$3x^2-6x+2\left(x^2-4\right)=4x^2-4x$$
$$3x^2-6x+2x^2-8=4x^2-4x$$
$$x^2-2x-8=0$$
$$(x-4)(x+2)=0$$
$$x=4,-2$$

Step 4: **Check all solutions**. You cannot assume that all solutions from step 3 are correct.

$x=4$ $$\frac{3}{4+2}+\frac{2}{4}=\frac{1}{2}+\frac{1}{2}=1$$
$$\frac{4\cdot 4-4}{4^2-4}=\frac{12}{12}=1$$ **OK**

$x=-2$, however, would cause division by zero if substituted into the original rational equation. Since this is undefined, it cannot be a solution.

Thus $x=4$ is the only solution.

4. Rational Functions and Equations

Q: We haven't worried about checking our solutions before – why was it necessary to check our solutions in this problem? How did we end up with a false "solution"?

A: Because of the multiplication in step 2. Similar to how we can "lose" solutions by dividing both sides of an equation by something containing a variable (i.e. when solving something like $x^2 = x$, simply dividing by x on both sides causes you to "lose" the solution $x = 0$), multiplying both sides of an equation by something containing a variable can sometimes "add" solutions that aren't actually there.

It's no coincidence that when we plug in these false solutions, we end up with undefined expressions. When we remove the fractions in a rational equation, numbers that aren't in the domain of the original rational expression can often sneak in as potential solutions.

Examples on the Board:

1. Solve $\dfrac{x}{x-5} - \dfrac{5}{x+5} = \dfrac{50}{x^2-25}$.

2. Solve $\dfrac{1}{3x+6} - \dfrac{1}{x^2-4} = \dfrac{3}{x-2}$