# 3. Polynomial Functions and Equations

I. Zeros of functions

**Def:** Given a function $f(x)$, a real number $c$ is a **zero** of $f$ if $f(c)=0$.

The following are equivalent:

1. Finding the zeros of $f$.
2. Solving $f(x)=0$.
3. Finding the x-coordinates of the x-intercepts of $f$.

II. Types of Polynomial Functions

Thus far, we've looked at constant, linear, and quadratic functions. These are all specific examples of *polynomial* functions.

**Def:** A **polynomial function** $P$ is given by

$$P(x)=a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0\ ,$$

where the coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers and the exponents are natural numbers. The first nonzero coefficient, $a_n$, is called the **leading coefficient**. Similarly, the term $a_n x^n$ is called the **leading term**. The **degree** of the polynomial is equal to the exponent $n$ in the leading term.

**Note:** An alternate way of defining constant, linear, and quadratic functions is in terms of degree. A constant function is a polynomial with degree 0; a linear function is a polynomial with degree 1; and so on. The following example lists the common names of the first several types of polynomials.

**Ex:**

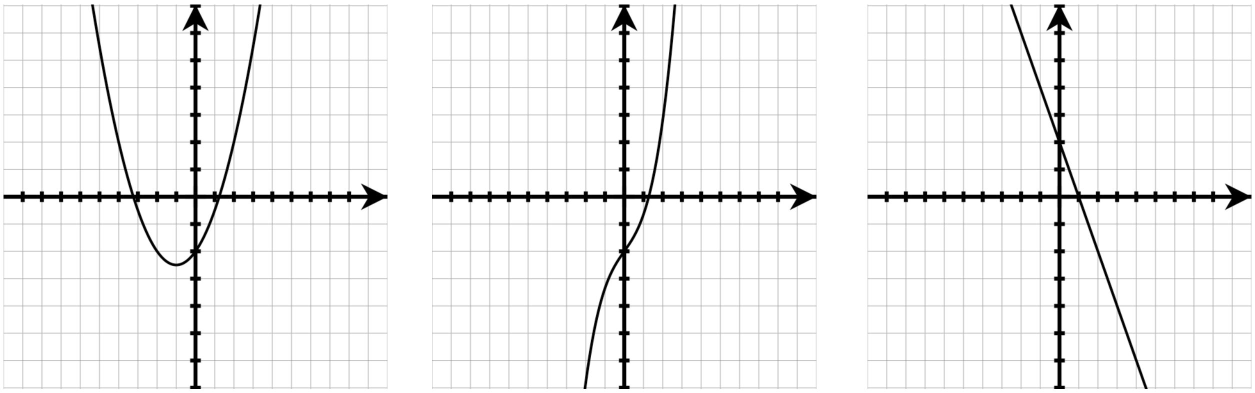
| | **type of polynomial** | **degree** | **leading coefficient** | **leading term** |
|---|---|---|---|---|
| $f(x)=3$ | constant | 0 | | |
| $f(x)=\frac{2}{3}x+1$ | linear | 1 | | |
| $f(x)=2x^2-x+3$ | quadratic | 2 | | |
| $f(x)=x^3+2x^2-5$ | cubic | 3 | | |
| $f(x)=-x^4+x^2-8$ | quartic | 4 | | |

# 3. Polynomial Functions and Equations

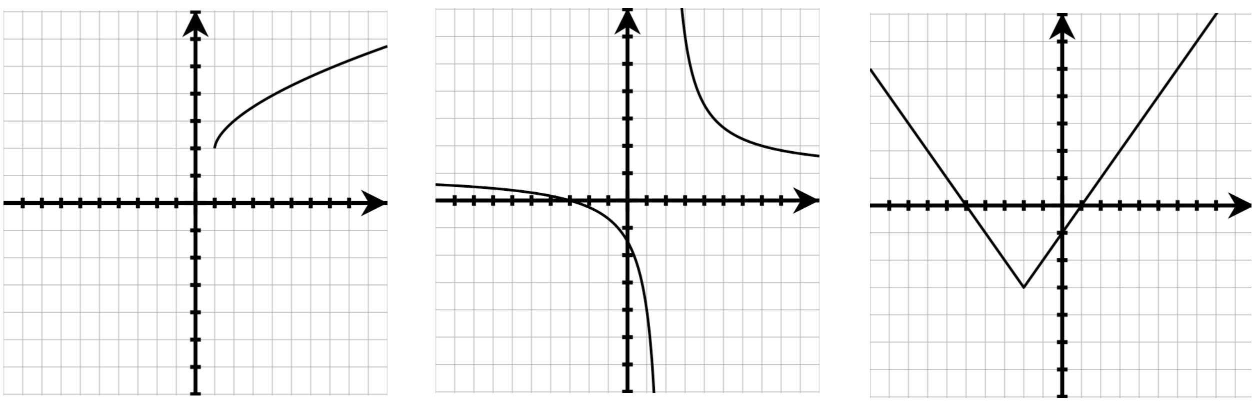
III. Characteristics of the Graphs of Polynomials

### 1. General characteristics

Look at the following graphs of sample polynomials.



Now look at the graphs of some sample functions that are definitely not polynomials.



You should notice a few characteristics that are shared by all three polynomial graphs but are not shared by the other three. In particular, there are three major characteristics that the graph of **any** polynomial will have:

1. The graph is **continuous** (there are no holes or breaks in the curve)
2. The graph is **smooth** (there are no sharp corners)
3. The graph is defined for all real numbers (the domain of any polynomial is $\mathbb{R}$)

**Q:** Why can't the bottom three graphs be polynomials? Find a reason for each.

**A:**

# 3. Polynomial Functions and Equations

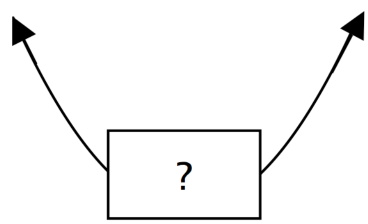
## 2. Limiting behavior

**Def:** The behavior of a polynomial as $x$ becomes very large ($x \to \infty$) or very negative ($x \to -\infty$) is referred to as the **limiting behavior** of the graph.

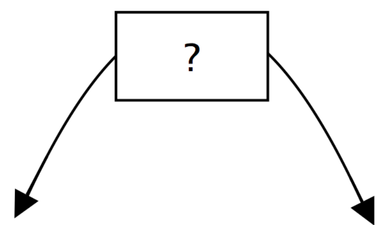
There are only four possible types of limiting behaviors for polynomials (not counting the trivial constant functions) and it is easy to determine which behavior an arbitrary polynomial exhibits. We do this with the Leading-Term Test.

**The Leading Term-Test:** If the leading term of a polynomial is $a_n x^n$, then the end behavior of the graph can be described in one of the four following ways:

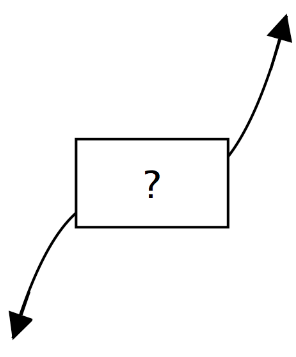
A. If $n$ is even, and $a_n > 0$:



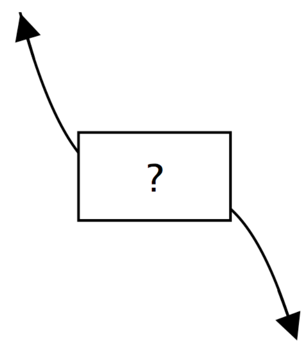

B. If $n$ is even, and $a_n < 0$:




C. If $n$ is odd, and $a_n > 0$:




D. If $n$ is odd, and $a_n < 0$:




**Note:** The leading-term test only determines end behavior, not the middle portion of the graph marked with a "?"

**Ex:** Match each function with the corresponding limiting behavior (A–D) above.

1. $-5x^3 + 2x^2 - x + 3$

2. $2x^4 + x^2 - 2x$

3. $-2x^2 + 4x^7 - 3 + x^5$

4. $-\frac{1}{8}x^{100} - 2x^3$

# 3. Polynomial Functions and Equations

## 3. x-intercepts

We can always find the x-intercepts of a graph of a function provided that we know that function's zeros (if $c$ is a real zero, then $(c,0)$ is an x-intercept). We've seen that for linear and quadratic functions finding all zeros is quite simple, but this is not the case for general polynomials.

**NEAT MATH FACT**
Though we have simple formulas for finding the zeros of linear and quadratic functions, the formulas for cubics and quartics are so complicated that they are almost never used. Moreover, it's been proven that no such formulas exist for solving general polynomials of degree larger than 4.

It is easy, however, to determine the zeros in certain cases - like when a polynomial has already been factored (simply use the zero product principle).

**Ex:** Find the zeros of $f(x) = 6(x-4)(x-7)^3(x+26)(x-2)^2$.

$$6(x-4)(x-7)^3(x+26)(x-2)^2 = 0$$
$$x = 4, \quad x = 7, \quad x = -26, \quad x = 2$$

In addition to making zeros (and thus x-intercepts) easy to find, a factored polynomial gives us information about the behavior of the graph at those x-intercepts when we consider something called **multiplicity.**

**Def:** Given a polynomial function , $f(x)$, and a zero of that function, $c$, then the **multiplicity** of $c$ is the number of times the factor $(x-c)$ is repeated in the factorization of $f(x)$. In other words, if $(x-c)^k$, where $k \geq 1$, appears in the fully simplified factorization of $f(x)$, then the multiplicity of $c$ is $k$.

**Ex:** Find the zeros and their multiplicities of $f(x) = -2(x+3)^2(x-5)^3(x+1)$.

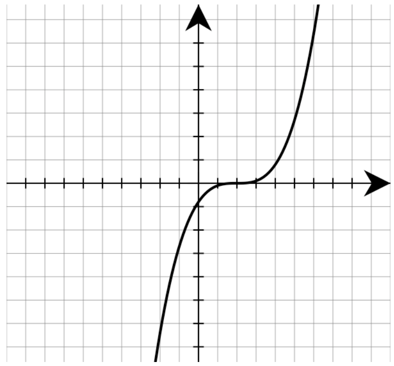
1. $-3$ is a zero with multiplicity $2$
2. $5$ is a zero with multiplicity $3$
3. $-1$ is a zero with multiplicity $1$

The multiplicity of a zero tells us the behavior of the function at the associated x-intercept.

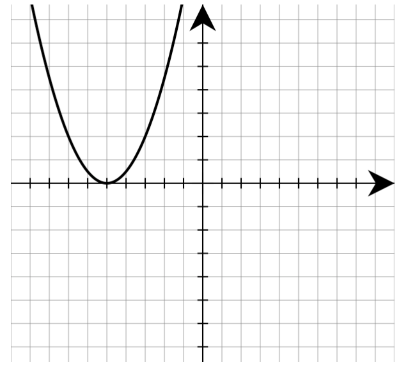
**Multiplicity Test:** Suppose a polynomial function $f(x)$ has a zero, $c$, with multiplicity $k$. Then:

1. If $k$ is odd, then the graph crosses the x-axis at $(c,0)$
2. If $k$ is even, then the graph is tangent to the x-axis at $(c,0)$

## 3. Polynomial Functions and Equations



1. graph crosses the x-axis



2. graph is tangent to the x-axis

The graph of a polynomial is **tangent** to the x-axis at $(c,0)$ if the graph has a relative max or min at $(c,0)$ (remember what a line tangent to a circle looks like from geometry?).

### 4. Polynomial facts

Every polynomial function of degree $n \geq 1$ has:

1. at most $n$ real zeros, so it has at most $n$ x-intercepts.
2. at most $n-1$ turning points (relative maximums or minimums)

IV. Polynomial Long Division and Synthetic Division

Consider the following problem (which you will never see on a test): $1075 \div 3$. In elementary school, you are asked to solve problems like this without a calculator. Remember long division?

$$\begin{array}{r} 358 \\ 3\,\big)\overline{1075} \\ \underline{9}\phantom{75} \\ 17\phantom{5} \\ \underline{15}\phantom{5} \\ 25 \\ \underline{24} \\ 1 \end{array}$$

which tells us that $\frac{1075}{3} = 358 + \frac{1}{3}$, or equivalently, $1075 = 3 \cdot 358 + 1$

The numbers involved all have special names:
$1075$ is the **dividend**
$3$ is the **divisor**
$358$ is the **quotient**
$1$ is the **remainder**

Why the remedial math? We're revisiting this simple technique because it is virtually identical to how we divide polynomials by polynomials – something many of you may not have done before. Just remember that it may look horrible with the x's, but it's essentially third grade math.

# 3. Polynomial Functions and Equations

We'll run through a sample problem step-by-step.

**Ex:** Divide $-3x^2 + x^3 + 24$ by $x - 1$

**Setup:** We set up polynomial division problems just as if we were doing ordinary long division – with two exceptions:

a. arrange the terms of both polynomials in descending order.

b. as you descend powers in the dividend, include any "missing" powers of $x$ using a coefficient of 0.

$$x-1 \overline{\smash{\big)}\, x^3 - 3x^2 + 0x + 24}$$

**Step 1:** Look at the first term in the dividend ($x^3$ in this case) and divide it by the first term in the divisor ($x$ in this case). Place the result above the first term in the dividend.

$$\begin{array}{r l}
 & x^2 \\
x-1 & \overline{\big)\, x^3 - 3x^2 + 0x + 24}
\end{array}$$

**Step 2:** Multiply the divisor by the previous step's result and place the resulting polynomial underneath the dividend. Be careful to line up terms with the same exponents. Subtract this polynomial from the terms directly above it.

$$\begin{array}{r l}
 & x^2 \\
x-1 & \overline{\big)\, x^3 - 3x^2 + 0x + 24} \\
 & \underline{x^3 - \phantom{3}x^2} \\
 & \phantom{x^3} -2x^2
\end{array}$$

**Step 3:** Bring down the next term from the divisor ($0x$ in this case) and repeat the preceding steps until there are no more terms to bring down.

$$\begin{array}{r l}
 & x^2 - 2x - 2 \\
x-1 & \overline{\big)\, x^3 - 3x^2 + 0x + 24} \\
 & \underline{x^3 - \phantom{3}x^2} \quad \downarrow \\
 & \phantom{x^3} -2x^2 + 0x \\
 & \phantom{x^3} \underline{-2x^2 + 2x} \quad \downarrow \\
 & \phantom{x^3 - 3x^2} -2x + 24 \\
 & \phantom{x^3 - 3x^2} \underline{-2x + \phantom{2}2} \\
 & \phantom{x^3 - 3x^2 + 0x + } 22
\end{array}$$

Just as with real numbers, this tells us that

$$\frac{x^3-3x^2+24}{x-1}=\left(x^2-2x-2\right)+\frac{22}{x-1} \quad or \quad x^3-3x^2+24=(x-1)\left(x^2-2x-2\right)+22$$

The quantities have the same names as before:

$x^3-3x^2+24$ is the **dividend**
$x-1$ is the **divisor**
$x^2-2x-2$ is the **quotient**
$22$ is the **remainder**

**Note:** From the $dividend = divisor \cdot quotient + remainder$ form, it should be clear that if we end up with a remainder of zero, then the divisor is actually a factor of the dividend.

**Example on the Board:**

Divide $\dfrac{2x^5-8x^4+38x^2+27}{x^2+x-3}$.

As we will see shortly, there are many useful applications for dividing polynomials by divisors of the form $x-c$. To that end, we will now learn a streamlined version of polynomial division called synthetic division.

# 3. Polynomial Functions and Equations

**Important Note:** Synthetic division **only** works for divisors of the form $x-c$.

**Ex:** Divide $-3x^2+x^3+24$ by $x-1$ (again).

**Setup:** If we're dividing by $x-c$, place $c$ in a box in the top left corner. **Notice the change in sign:** we're dividing by $x-1$, but we place $1$ (not $-1$) in the box. To the right of the box, list the coefficients of the terms in the dividend in decreasing order. As before we want to include any "missing" powers of $x$ using a $0$ coefficient. Then draw the lines as shown.

$$\begin{array}{r|rrrr} 1 & 1 & -3 & 0 & 24 \\ & & & & \\ \hline & & & & \end{array}$$

**Step 1:** Drop the first coefficient in the list (not the boxed one).

$$\begin{array}{r|rrrr} 1 & 1 & -3 & 0 & 24 \\ & \downarrow & & & \\ \hline & 1 & & & \end{array}$$

**Step 2:** Multiply the number under the line ($1$ in this case) by the number in the box, and place it underneath the next coefficient in the list.

$$\begin{array}{r|rrrr} 1 & 1 & -3 & 0 & 24 \\ & & 1 & & \\ \hline & 1 & & & \end{array}$$

**Step 3:** Add the two numbers in that column ($-3$ and $1$ in this case) and place the result beneath the line.

$$\begin{array}{r|rrrr} 1 & 1 & -3 & 0 & 24 \\ & & 1 & & \\ \hline & 1 & -2 & & \end{array}$$

**Step 4:** Repeat steps 2 and 3, until you reach the end of the list of coefficients.

$$\begin{array}{r|rrr|r} 1 & 1 & -3 & 0 & 24 \\ & & 1 & -2 & -2 \\ \hline & 1 & -2 & -2 & 22 \end{array}$$

The numbers along the bottom (to the left of the vertical line) are the coefficients of the quotient, $x^2-2x-2$, and the number to the right of the vertical line is the remainder, $22$.

# 3. Polynomial Functions and Equations

**Examples on the board:**

1. Use synthetic division to divide $\dfrac{x^4 - 4x^3 - 7x^2 + 12x - 7}{x + 2}$.

2. Use synthetic division to divide $\dfrac{x^4 + 11x^3 + 41x^2 + 61x + 30}{x + 3}$.

3. Substitute $-2$ into $x^4 - 4x^3 - 7x^2 + 12x - 7$.

4. Substitute $-3$ into $x^4 + 11x^3 + 41x^2 + 61x + 30$.

## V. Remainder and Factor Theorems

Isn't it curious that when you plug $-2$ into $x^4 - 4x^3 - 7x^2 + 12x - 7$, you end up with the remainder from dividing $x^4 - 4x^3 - 7x^2 + 12x - 7$ by $x - (-2)$? This is not a coincidence. It is our first major use for synthetic division.

**The Remainder Theorem:** If $c$ is a number and $f(x)$ is a polynomial, then $f(c)$ is equal to the remainder obtained when dividing $f(x)$ by $x - c$.

What this means is that synthetic division can actually be used to evaluate a polynomial at a value without having to worry about horrible calculations involving exponents. It also allows us to check whether a given number is a zero of a polynomial with relative ease.

**Examples on the board:**

1. Use synthetic division to find $f(5)$ where $f(x) = x^5 - 10x^4 + 20x^3 + 115x + 100$ .

2. Use synthetic division to see if $-4$ is a zero of $f(x) = 3x^3 + 11x^2 - 2x + 8$ .

We now consider a result that follows from the Remainder Theorem.

**Factor Theorem:** For a polynomial $f(x)$, if $f(c) = 0$, then $x - c$ is a factor of $f(x)$.

This means that we can use synthetic division to help us factor larger polynomials provided that we make some educated guesses as to what potential factors might be.

VI. Facts about the Zeros of Polynomials

So far in this course, we've only worked with polynomials having real coefficients (primarily integer coefficients at that). However, the next theorem is true even for polynomials with complex coefficients. It is the first major result from abstract algebra that we will discuss.

**The Fundamental Theorem of Algebra:**
Every polynomial function $f(x)$ of degree $n \geq 1$ has exactly $n$ complex zeros when we count the multiplicity of any repeated zeros.

# 3. Polynomial Functions and Equations

## 1. Finding Polynomials with Given Zeros

Since we know the connection between the zeros of a polynomial and it's factors (if $c$ is a zero, then $x-c$ is a factor, and vice versa), we can construct polynomials that have a given set of zeros .

**Ex:** Find a polynomial function of degree 3 with zeros $-3$, $0$, and $\frac{1}{2}$.

We know it has factors $(x+3)$, $(x-0)$, and $\left(x-\frac{1}{2}\right)$.

Then the simplest function of degree 3 would be as follows:

$$\begin{aligned} f(x) &= x(x+3)\left(x-\frac{1}{2}\right) \\ &= x\left(x^2-\frac{1}{2}x+3x-\frac{3}{2}\right) \\ &= x\left(x^2+\frac{5}{2}x-\frac{3}{2}\right) \\ &= x^3+\frac{5}{2}x^2-\frac{3}{2}x \end{aligned}$$

**Example on the Board:**

Find a polynomial function of degree 5 with zeros $-2$ (mult. 2) and $0$ (mult. 3).

## 2. Theorems about the Zeros of Polynomials

Have you noticed that whenever we solve quadratic equations, if there is an imaginary solution then its conjugate is also a solution? This doesn't just happen in quadratics, but in any polynomial with real coefficients.

**Imaginary Zeros Theorem:** If $a+bi$, where $a$ and $c$ are real numbers and $b \neq 0$, is a zero of a polynomial with real coefficients, then so is $a-bi$.

Similarly, you may have noticed that when we've solved quadratic equations, if there is a solution involving a square root, then its conjugate is also a solution. This occurs in any polynomial with rational coefficients.

**Irrational Zeros Theorem:** If $a+c\sqrt{b}$, where $a$ and $c$ are rational numbers and $b$ is not a perfect square (1, 4, 9, 16, ...), is a zero of a polynomial with rational coefficients, then so is $a-c\sqrt{b}$.

# 3. Polynomial Functions and Equations

**Examples on the Board:**

1. Suppose that a polynomial function of degree 5 with rational coefficients has zeros $-3-3i$, $2+\sqrt{13}$, and $6$. Find all other zero(s).

2. Find a polynomial of lowest degree with rational coefficients that has zeros $-\sqrt{2}$ and $4i$.

As opposed to the previous two theorems, the next theorem doesn't require that you already know a particular zero in order to obtain a new one. In fact, the next theorem (Rational Zeros Theorem) actually gives you a list of all possible rational zeros of a polynomial. For that reason, it is an extremely powerful tool in searching for zeros, and hence factors, of polynomials. Unfortunately, there is a further restriction placed on the polynomial coefficients – for the Rational Zeros Theorem to apply we now need integer coefficients.

**Rational Zeros Theorem:**

Let $f(x)=a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial with integer coefficients. Consider a reduced rational number $\frac{p}{q}$ ($p$ and $q$ have no common factors except $\pm 1$). If $\frac{p}{q}$ is a zero of $f(x)$, then $p$ is a factor of $a_0$ and $q$ is a factor of $a_n$.

Now we can narrow down the number of possible rational zeros for any polynomial with integer coefficients to a finite list! In combination with the factor theorem, we can use synthetic division to check all the possible rational zeros very quickly. We now have a powerful tool to help us factor integer polynomials and also to find x-intercepts.

**Ex:** For $f(x)=3x^4-4x^3+x^2+6x-2$, find the rational zeros (and then the other zeros). Then factor $f(x)$.

**Step 1:** Find all possible rational zeros using the Rational Zeros Theorem.

Factors of $-2$ are $\pm 1$ and $\pm 2$.
Factors of $3$ are $\pm 1$ and $\pm 3$.
Thus the possible rational zeros are $\pm 1$, $\pm 2$, $\pm\frac{1}{3}$, and $\pm\frac{2}{3}$.

**Step 2:** Use synthetic division to check the possible rational zeros.

Try $x = 1$

$$\begin{array}{r|rrrrr} 1 & 3 & -4 & 1 & 6 & -2 \\ & & 3 & -1 & 0 & 6 \\ \hline & 3 & -1 & 0 & 6 & |4 \end{array}$$

Not a zero.

Try $x = -1$

$$\begin{array}{r|rrrrr} -1 & 3 & -4 & 1 & 6 & -2 \\ & & -3 & 7 & -8 & 2 \\ \hline & 3 & -7 & 8 & -2 & |0 \end{array}$$

-1 is a zero!

**Step 3:** Factor $x+1$ out of $f(x)$ (can read off the coefficients from last step).

$$f(x) = (x+1)\left(3x^3 - 7x^2 + 8x - 2\right)$$

**Step 4:** Now repeat the procedure on the quotient $3x^3 - 7x^2 + 8x - 2$. It's smaller, so the synthetic division will go quicker.

The coefficients are the same as before, so we have the same possible rational zeros: $\pm 1$, $\pm 2$, $\pm\frac{1}{3}$, and $\pm\frac{2}{3}$.

We would then check each of these in synthetic division. For the sake of brevity, I'll skip ahead and guess that I should check $\frac{1}{3}$ first.

$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & -7 & 8 & -2 \\ & & 1 & -2 & 2 \\ \hline & 3 & -6 & 6 & |0 \end{array}$$

what a guess!

Factor out $x - \frac{1}{3}$ as before.

$$f(x) = (x+1)\left(x - \frac{1}{3}\right)\left(3x^2 - 6x + 6\right)$$

**Step 5:** Now we only have a quadratic left, which we can solve normally.

Factor out the 3.

$$f(x)=3(x+1)\left(x-\frac{1}{3}\right)(x^2-2x+2)$$

Use the quadratic formula on $x^2-2x+2$.

$$\begin{aligned}x&=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\&=\frac{2\pm\sqrt{(-2)^2-4(1)(2)}}{2(1)}\\&=\frac{2\pm\sqrt{-4}}{2}\\&=1\pm i\end{aligned}$$

Thus the zeros are $-1$, $\frac{1}{3}$, and $1\pm i$.
The factoring of $f$ is then $f(x)=3(x+1)(x-\tfrac{1}{3})(x-(1-i))(x-(1+i))$.

**Examples on the Board:**

1. For $f(x)=x^3-x^2-3x+3$, find the rational zeros (and then the other zeros). Then factor $f(x)$.

2. For $f(x)=2x^3+7x^2+2x-8$, find the rational zeros (and then the other zeros). Then factor $f(x)$.