2. Quadratic Equations

Zero Product Principle: If $ab = 0$, then $a = 0$ or $b = 0$.

I. Solving Quadratic Equations

DEF A **quadratic equation** is one that can be written in the form $ax^2 + bx + c = 0$ where a, b, and c are real numbers and $a \neq 0$. This is called the **standard form** of a quadratic equation.

There are three general ways to solve a quadratic equation:

1). Factoring (doesn't work for all)
2). Completing the square
3). Quadratic formula

} You should be able to do all three!

A. Solving by Factoring:

Consider the quadratic equation $2x^2 + 7x = 15$.

Step 1: Get it in standard form ($ax^2 + bx + c = 0$).

$$2x^2 + 7x - 15 = 0$$

Step 2: Factor.

$$(2x - 3)(x + 5) = 0$$

Step 3: Apply the Principle of Zero Products.

$$2x - 3 = 0 \quad \text{or} \quad x + 5 = 0$$

$$x = \boxed{\frac{3}{2}} \qquad\qquad x = \boxed{-5}$$

In practice we only solve by factoring when it is easy to do so. The quadratic formula will be much more powerful, but it also tends to be more of a hassle.

B. Solving by Completing the Square:

Consider the quadratic equation $3x^2 + 5 = -18x$.

Step 1: Rewrite the equation in standard form ($ax^2 + bx + c = 0$).

$$3x^2 + 18x + 5 = 0$$

Step 2: Complete the square on the quadratic function.

$$3\left(x^2+6x\right)+5=0$$
$$3\left(x^2+6x+9\right)+5-3\cdot 9=0$$
$$3\left(x+3\right)^2-22=0$$

Step 3: Isolate the squared term on one side of the equation.

$$3\left(x+3\right)^2-22=0$$
$$3\left(x+3\right)^2=22$$
$$\left(x+3\right)^2=\frac{22}{3}$$

Step 4: Take the square root of both sides and solve for x.

$$\sqrt{\left(x+3\right)^2}=\sqrt{\frac{22}{3}}$$
$$\left|x+3\right|=\sqrt{\frac{22}{3}}$$
$$x+3=\pm\sqrt{\frac{22}{3}}$$
$$x=\boxed{\pm\sqrt{\frac{22}{3}}-3}$$

C. Solving by Quadratic Formula:

Since completing the square is a procedure that can be applied to solve any quadratic, what happens if we use it solve an arbitrary quadratic equation $ax^2+bx+c=0$? You derive what is known as the quadratic formula.

Quadratic Formula: The solutions of $ax^2+bx+c=0$, where $a\neq 0$, are given by

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Ex: Rework the problem we just solved by completing the square: $3x^2+5=-18x$.

Step 1: Rewrite the equation in standard form ($ax^2+bx+c=0$).

$$3x^2+18x+5=0$$

2. Quadratic Equations

Step 2: Apply the quadratic formula.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-18 \pm \sqrt{18^2 - 4(3)(5)}}{2(3)} \\ &= \frac{-18 \pm \sqrt{264}}{6} \\ &= \boxed{-3 \pm \frac{\sqrt{66}}{3}} \end{aligned}$$

Examples on the board:

1. Solve $5x^2 = 12x$ by factoring.

2. Solve $3x^2 - 6x + 10 = 0$ by completing the square.

3. Solve $4x^2 + 1 = 4x$ using the quadratic formula.

2. Quadratic Equations

II. Number and Type of Solutions of Quadratic Equations

In the examples thus far we have seen that both the number and type of solutions of quadratic equations can vary. In fact, there are exactly three possible types of solutions:

1. two distinct real solutions (see previous example #1)
2. two distinct imaginary solutions (always complex conjugates; see previous example #2)
3. one real solution (also known as a repeated solution; see previous example #3)

It is actually possible to determine which type of solution a quadratic equation will have without explicitly solving the equation. We do so using the portion of the quadratic formula referred to as the discriminant.

Def: The **discriminant** of $ax^2 + bx + c = 0$, where $a \neq 0$, is the quantity $b^2 - 4ac$.

Theorem: For $ax^2 + bx + c = 0$, where $a \neq 0$:

- if $b^2 - 4ac = 0$, then the equation has one real solution.
- if $b^2 - 4ac > 0$, then the equation has two distinct real solutions.
- if $b^2 - 4ac < 0$, then the equation has two distinct imaginary solutions.

Examples on the board: Determine the number and type of solutions using the discriminant.

1. $4x^2 + x + 12 = 0$

2. $2x^2 - 6x + 1 = 0$

3. $4x^2 + 12x + 9 = 0$

III. "Reducible to Quadratic" Equations

Some very special non-quadratic equations can be solved using quadratic methods by making a substitution of variables. These types of equations are said to be **reducible to quadratic**.

Ex: Solve $x^6 - 7x^3 - 8 = 0$.

While not a quadratic equation, this is reducible to quadratic in the following sense:
if we were to think of our variable as being x^3 instead of x, we could rewrite the equation as a quadratic (using this new variable $u = x^3$).

2. Quadratic Equations

Step 1: Notice that if we were to make the substitution $u = x^3$, the resulting equation would be a quadratic equation with variable u.

$$x^6 - 7x^3 - 8 = 0$$
$$\left(\boxed{x^3}\right)^2 - 7\boxed{x^3} - 8 = 0$$
$$u^2 - 7u - 8 = 0$$

Step 2: Now we solve this quadratic for our new variable u:

$$u^2 - 7u - 8 = 0$$
$$(u-8)(u+1) = 0$$
$$u = 8, -1$$

Step 3: Finally, we substitute x back into the equation and solve for x:

$$u = 8 \qquad u = -1$$
$$x^3 = 8 \;,\; x^3 = -1$$
$$x = \boxed{2} \qquad x = \boxed{-1}$$

Examples on the board:

1. Solve $x^4 + 3 = 4x^2$

2. Solve $2x + 5\sqrt{x} + 1 = 0$