# 1. The Complex Numbers

## I. The Complex Numbers

**Def:** The **number** $i$ is defined such that $i^2 = -1$. In other words, $i = \sqrt{-1}$.

**Examples on the board:** Express each number in terms of $i$.

1. $\sqrt{-81}$

2. $-\sqrt{-3}$

3. $\sqrt{-75}$

4. $\left(\sqrt{-4}\right)\left(\sqrt{-36}\right)$

**Def:** Numbers of the form $a+bi$, where $a$ and $b$ are real numbers, are called **complex numbers** (aka imaginary numbers). Typically. we refer to $a$ as the **real part** of $a+bi$ and to $b$ as the **imaginary part** of $a+bi$.

**Notation:** The set of all complex numbers is denoted by $\mathbb{C}$.

**Q:** True or false: $\mathbb{R} \subset \mathbb{C}$?

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II. Addition/Subtraction/Multiplication with Complex Numbers

Addition and subtraction with complex numbers is pretty straightforward. You just add/subtract the real parts together and add/subtract the imaginary parts together (i.e. treat $i$ like a variable and add/subtract like terms).

**Examples on the board:** Simplify.

1. $(-6-5i)+(9+2i)$

2. $(8-3i)-(9-i)$

Multiplication of complex numbers is also fairly easy. As you might expect, you just use FOIL to expand the multiplication and then simplify the result.

**Example on the board:**

3. $(3-2i)(-7+2i)$

4. $\left(6+\sqrt{-5}\right)^2$

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III. Simplifying large powers of $i$

Notice that there seems to be a pattern to the values of the powers of $i$ (shown below).

$$i^0 = 1$$
$$i^1 = i$$
$$i^2 = -1$$
$$i^3 = -i$$
$$i^4 = 1$$
$$i^5 = i$$
$$\vdots$$

In particular, the successive powers of $i$ will cycle through the values $i$, -1, -$i$, and 1. We can use this to our advantage and simplify large powers of $i$ very easily. All we have to know are the first four values on the list above ($i^0 = 1$, $i^1 = i$, $i^2 = -1$, and $i^3 = -i$).

**Ex:** Simplify $i^{75}$.

Find the remainder when you divide the power of $i$ by 4 (in this example, $75 \div 4$ has remainder 3). Then $i$ to the original power ($i^{75}$ in this case) is equal to $i$ with exponent equal to the remainder you just calculated ($i^3$ in this case).

Since $75 \div 4$ has remainder 3, $i^{75} = i^3 = -i$

**Examples on the board:** Simplify

1. $i^{22}$

2. $i^{41}$

3. $i^{100}$

4. $(2i)^5$

5. $(-i)^{53}$

IV. Conjugates and Division of Complex Numbers

**Def:** The **complex conjugate** of a complex number $a+bi$ is $a-bi$.

Why do we care about conjugates? It turns out that the product of a complex number and its complex conjugate is always a real number. This will actually allow us to talk about division with complex numbers.

**Examples on the board:** Find the complex conjugate of each number.

1. $-10-i$

2. $14i$

3. $25$

We use conjugates in the division of complex numbers in a very similar way that we rationalize the denominator of a function with roots in the denominator.

**Ex:** Simplify $\dfrac{5-i}{-7+2i}$.

**Step 1:** If necessary, simplify the fraction so that it is of the form $\dfrac{a+bi}{c+di}$.

$\dfrac{5-i}{-7+2i}$ is already in this form, so we don't need to do anything.

**Step 2:** Multiply numerator and denominator by the complex conjugate of the denominator and simplify.

$$\begin{aligned}\frac{5-i}{-7+2i} &= \left(\frac{5-i}{-7+2i}\right)\left(\frac{-7-2i}{-7-2i}\right)\\ &= \frac{-35-10i+7i+2i^2}{49+14i-14i-4i^2}\\ &= \frac{-35-3i+2(-1)}{49-4(-1)}\\ &= \frac{-37-3i}{53}\end{aligned}$$

**Step 3:** Separate the fraction into it's real and imaginary parts. You need to end up with a complex number of the form $a+bi$ where $a$ and $b$ are **real numbers**.

$$\frac{-37-3i}{53} = -\frac{37}{53} - \frac{3}{53}i$$

**Examples on the board:** Simplify.

1. $\dfrac{i}{3-5i}$

2. $\dfrac{2-i}{(4-i)^2}$